

RATIONALITY PROBLEM OF CONIC BUNDLES

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ABSTRACT. Let k be a field with $\text{char } k \neq 2$, X be an affine surface defined by the equation $z^2 = P(x)y^2 + Q(x)$ where $P(x), Q(x) \in k[x]$ are separable polynomials. We will investigate the rationality problem of X in terms of the polynomials $P(x)$ and $Q(x)$. The rationality of the conic bundle X over \mathbb{P}_k^1 was studied by Iskovskikh [Isk67], [Isk70], [Isk72], but he formulated his results in geometric language. This paper aims to give an algebraic counterpart.

1. INTRODUCTION

Throughout this paper, k is a field with $\text{char } k \neq 2$. It is not assumed that k is algebraically closed; in fact, the most interesting results of this paper is the case when k is a non-closed field.

Let K be a field extension of k . We will say that K is k -rational if K is isomorphic to the rational function field $k(x_1, x_2, \dots, x_n)$ over k with variables x_1, \dots, x_n for some positive integer n . An irreducible algebraic variety X defined over k is called k -rational if its function field $k(X)$ is k -rational.

Iskovskikh studied the rationality of conic bundles and obtained the following result [Isk67], [Isk70], [Isk72].

Theorem 1.1 (Iskovskikh). *Let X be a fibred rational k -surface as a standard conic bundle $\pi : X \rightarrow \mathbb{P}_k^1$. If X has at least four degenerate geometric fibres, then X is not k -rational.*

The function field of such a conic bundle is isomorphic to $k(x, y, z)$ with the relation

$$(1.1) \quad z^2 = Q(x)y^2 + P(x), \quad P, Q \in k[x],$$

where $k(x, y)$ is the rational function field over k with two variables x, y .

Thus Iskovskikh's theorem (Theorem 1.1) is equivalent to the rationality problem of the field $K := k(x, y, z)$ with the relation defined by (1.1). In this paper, we will give a necessary and sufficient condition for the rationality of K in terms of the polynomials P and Q , assuming that both P and Q are separable polynomials. In this sense, our results may be regarded as an algebraic counterpart of Iskovskikh's theorem.

In Subsection 1.1 and Section 3 of this paper, we will consider the case where $\deg Q(x) = 0$, i.e. $Q(x) = a \in k$. The case where $\deg Q(x) \geq 1$ will be discussed in Subsection 1.2 and Section 4.

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1.1. Main result of generalized Châtelet surfaces. First of all, let $K = k(x, y, z)$ be a field defined by the equation

$$(1.2) \quad z^2 = ay^2 + P(x), \quad a \in k, P(x) \in k[x].$$

Remark 1.1. The surface X defined by (1.2) is called a Châtelet surface when $\deg P = 3$ or 4 , which was studied by Châtelet [Châ59]. Thus we will call the surface X a generalized Châtelet surface when P is any non-zero polynomial in $k[x]$. The function field of X is the field K defined by (1.2).

Let K be the function field of a generalized Châtelet surface defined by the equation (1.2). Note that

- (a) If $a \in k^2$, then K is k -rational.

When $\sqrt{a} \in k$, define $u = z + \sqrt{a}y$ and $v = z - \sqrt{a}y$. (1.2) becomes $uv = P(x)$; thus $K = k(x, y, z) = k(x, u, v) = k(x, u)$ since $v = \frac{P(x)}{u} \in k(x, u)$. From now on, we will assume that $\sqrt{a} \notin k$.

- (b) Obviously we may assume that P contains no multiple irreducible factor in $k[x]$.

When $\deg P = 1$, (1.2) is written as $z^2 = ay^2 + x$ so $K = k(x, y, z) = k(y, z)$ is k -rational.

When $\deg P = 0$, (1.2) is written as $z^2 = ay^2 + b$. Then $K = k(x, y, z)$ is k -rational if and only if the quadratic form $aY^2 + bX^2 = Z^2$ has a non-trivial zero over k , i.e. the norm residue symbol of degree two $(a, b)_{2,k} = 0$.

When $\deg P = 2$ and $\text{char } k \neq 2$, (1.2) may be written as $z^2 = ay^2 + bx^2 + c$. If $c \neq 0$, then $K = k(x, y, z)$ is k -rational if and only if $c \in k^2 - ak^2 - bk^2$. If $c = 0$, as before, K is k -rational if and only if $(a, b)_{2,k} = 0$. See Theorem 6.7 of [HKO94] for details.

- (c) Let l be the splitting field of $P(x)$. If $\deg P \geq 3$ and $l \cap k(\sqrt{a}) = k$, then K is not k -rational by a rationality criterion of Manin [Man67], which will be explained in Subsection 3.4 to Subsection 3.6.
- (d) Suppose that some irreducible component P_1 of P is of the form $P_1(x) = A(x)^2 - aB(x)^2$ where $A(x), B(x) \in k[x]$. Define $z = A(x)z' + aB(x)y'$ and $y = B(x)z' + A(x)y'$. We have $z^2 - ay^2 = P_1(x)(z'^2 - ay'^2)$. It follows that $z'^2 - ay'^2 = P(x)/P_1(x)$. Since $K = k(x, y, z) = k(x, y', z')$, the rationality of $k(x, y, z)$ does not change if we replace P by P/P_1 .

From the above discussion, we may assume the following conditions without loss of generality.

- (C1) $a \notin k^2$.
- (C2) $\deg P \geq 3$ and $P \in k[x]$ is square-free.
- (C3) If l is the splitting field of $P(x)$, then $k(\sqrt{a}) \subset l$.
- (C4) Every irreducible factor of $P(x)$ is also irreducible over $k(\sqrt{a})$, which is equivalent to that no irreducible factor of $P(x)$ in $k[x]$ is of the form $A(x)^2 - aB(x)^2$.
- (C5) $\text{char } k \neq 2$, and every irreducible factor of $P(x)$ is separable over k ; this is the assumption prescribed at the beginning of this paper.

Our main result is:

Theorem 1.2. *The field $K = k(x, y, z)$ defined by (1.2) is not k -rational under the assumptions (C1), ..., (C5).*

Remark 1.2. The non-rationality of K for the case where $\deg P = 3$ (and for some other cases) is proved by V. A. Iskovskikh and B. E. Kunyavskii (see [Kan07, Theorem 4.1]). The case where $\deg P = 3$ and $P(x)$ is irreducible is a typical example of a surface which is not k -rational but stably k -rational (see Beauville, Colliot-Thélène, Sansuc and Swinnerton-Dyer [BCSS85]).

1.2. Main result of conic bundles. Now we shall deal with the rationality of a general conic bundle, whose function field is $K := k(x, y, z)$ satisfying

$$(1.3) \quad z^2 = P(x)y^2 + Q(x)$$

where $P, Q \in k[x]$ are separable polynomials and $\deg P \geq 1$, $\deg Q \geq 1$. Remember that k is a field with $\text{char } k \neq 2$.

As before, the same problem was studied by Iskovskikh [Isk67, Isk70, Isk72] as the rationality of standard conic bundles. Our approach is essentially an adaptation of Iskovskikh's idea, but we will give a necessary and sufficient condition for the rationality in terms of P and Q explicitly as below.

Let $s = s_1 + s_2 + s_3 + s_4$, where s_1 (resp. s_2 , resp. s_3) is the number of $c \in \bar{k}$ such that $P(c) = 0$ and $Q(c) \notin k(c)^2$ (resp. $Q(c) = 0$ and $P(c) \notin k(c)^2$, resp. $P(c) = Q(c) = 0$ and $-\frac{Q}{P}(c) \notin k(c)^2$). $s_4 = 0$ or 1 and $s_4 = 1$ if and only if one of the following three conditions are satisfied:

- (i) $\deg P$ is even, $\deg Q$ odd and $p_0 \notin k^2$;
- (ii) $\deg P$ is odd, $\deg Q$ even and $q_0 \notin k^2$;
- (iii) $\deg P$ is odd, $\deg Q$ odd and $-q_0/p_0 \notin k^2$.

Here p_0 (resp. q_0) is the coefficient of the highest degree term of P (resp. Q).

Our main result is:

Theorem 1.3. *Let $K = k(x, y, z)$ be the field defined by (1.3).*

- (1) *When $s \geq 4$, $k(x, y, z)$ is not k -rational.*
- (2) *When $s = 3$, $k(x, y, z)$ is k -rational.*
- (3) *The case $s = 1$ can not happen.*
- (4) *When $s = 0$ or $s = 2$, $k(x, y, z)$ is not k -rational if and only if (I) both of $\deg P$ and $\deg Q$ are even and, (II) $a^2p_0 + b^2q_0 = c^2$ has no non-zero solution (a, b, c) in k (resp. $k(\sqrt{\pi_1})$) for $s = 0$ (resp. $s = 2$). When $s = 2$, π_1 satisfies one of the following three conditions: (i) $P(c_1) = 0$ and $Q(c_1) = \pi_1 \notin k^2$; (ii) $Q(c_1) = 0$ and $P(c_1) = \pi_1 \notin k^2$; (iii) $P(c_1) = Q(c_1) = 0$ and $-\frac{Q}{P}(c_1) = \pi_1 \notin k^2$.*

Remark 1.3. As for the case $s = 2$ in Theorem 1.3 (4), if $c_1, c_2 \in k$ then $k(x, y, z)$ is k -rational, otherwise c_1 and c_2 are k -conjugate and we can show that $\pi_1 \in k$ (so $k_1 = k(\sqrt{\pi_1})$ is a quadratic extension of k).

Remark 1.4. The non-rationality for the case where $P(x) = x$ and $Q(x) = f(x^2)$ is discussed by B. E. Kunyavskii, A. N. Skorobogatov and M. A. Tsfasman (see [KST89, Chapter 6]).

1.3. Ideas of the proof. The field $\bar{k}(x, y, z)$ is $\bar{k}(x)$ -rational, i.e. $\bar{k}(x, y, z) = \bar{k}(x, u)$ for some $u \in \bar{k}(x, y, z)$. The action of $\mathfrak{G} = \text{Gal}(k^{\text{sep}}/k)$ on u induces birational transformation of $\mathbb{P}^1 \times \mathbb{P}^1$. After finite steps of blowings-up and down of $\mathbb{P}^1 \times \mathbb{P}^1$, these birational transformations become biregular on a surface X . Then, the group action of \mathfrak{G} induces a permutation of irreducible curves. Thus the divisor group $\text{Div}(X)$ becomes a permutation \mathfrak{G} -module, i.e. $\text{Div}(X)$ has a \mathbb{Z} -basis

permuted by \mathfrak{G} . Since the principal divisor group is stable under the action of \mathfrak{G} , the Picard group $\text{Pic}(X)$ is also a \mathfrak{G} -module.

From the structure of $\text{Pic}(X)$ as a \mathfrak{G} -module, we will derive the k -irrationality of K . Three criteria will be instrumental in our proof. We list them in the following.

I. Non-triviality of $H^1(\mathfrak{G}, \text{Pic}(X))$.

The first Galois cohomology $H^1(\mathfrak{G}, \text{Pic}(X))$ is k -birational invariant (see [Man69, pages 150-151, Theorem 2.2, Corollary 2.3]). In particular, if K is k -rational, then $H^1(\mathfrak{G}, \text{Pic}(X)) = 0$.

The following theorem for a generalized Châtelet surface is due to Sansuc [San81, Proposition 1 (v)] (see also [CTS94], [Sko01, Proposition 7.1.1]). For the convenience of the reader, we will give a proof of it in Subsection 3.6.

Theorem 1.4 (Sansuc [San81]). *Let r' be the number of irreducible components of $P(x)$ over k . Define j by $j = r' - 1$ if $\deg P$ is odd; define $j = r' - 1$, if $\deg P$ is even and every irreducible component of P is of even degree; define $j = r' - 2$, if $\deg P$ is even and some irreducible component of P is of odd degree. Then $H^1(\mathfrak{G}, \text{Pic}(X)) = (\mathbb{Z}/2\mathbb{Z})^j$.*

Theorem 1.4 implies that K is not k -rational except when $P(x)$ is irreducible, or a product of two irreducible polynomials of odd degree.

II. Calculating the intersection form.

If X is birational to $\mathbb{P}^1 \times \mathbb{P}^1$ over k , there exist two families of \mathfrak{G} -invariant irreducible curves $\{C_a\}$ and $\{C'_a\}$ on X , parametrized by elements of k .

After successive blowings-up at fundamental points of $\mathbb{P}^1 \times \mathbb{P}^1$ and X respectively, we will obtain surfaces Z and Z' which are biregular over k . Except finite number of elements of k , C_a and C'_a (denoted by C for simplicity) satisfy the conditions that $C \cdot C = 0$ and $C \cdot \Omega = -2$ on Z' , where Ω is the canonical divisor.

By a blowing-up E_j , $C \cdot C$ decreases by $(C \cdot E_j)^2$ and $C \cdot \Omega$ increases by $C \cdot E_j$, so we must have

$$(1.4) \quad C \cdot C = \sum_j m_j^2, \quad C \cdot \Omega = -2 - \sum_j m_j$$

on X , where $m_j = C \cdot E_j$.

On the other hand, we will prove

Theorem 1.5.

(1) *If $K = k(x, y, z)$ is a generalized Châtelet surface defined by (1.2) and $\deg P \geq 7$, then K is not k -rational.*

(2) *If $K = k(x, y, z)$ is a general conic bundle defined by (1.3) and $s \geq 8$ (s is as in the last paragraph before Theorem 1.3), then K is not k -rational.*

In fact, there is a non-singular projective surface Y which is birational to X , such that any \mathfrak{G} -invariant irreducible curve C other than $x = \text{const.}$ will not satisfy (1.4) for any further blowing-up $\{E_j\}$.

III. Reduction to a del Pezzo surface.

A del Pezzo surface S is biregular to some successive blowings-up of the projective plane \mathbb{P}^2 .

Theorem 1.6.

- (1) If $K = k(x, y, z)$ is a generalized Châtelet surface defined by (1.2) and $3 \leq \deg P \leq 6$, then K is not k -rational.
- (2) If $K = k(x, y, z)$ is a general conic bundle defined by (1.3) and $4 \leq s \leq 7$ (s is as in the last paragraph before Theorem 1.3), then K is not k -rational.

In fact, suppose K is k -rational, then there is a del Pezzo surface X'' which is birational to X . Thus, X'' is biregular to some successive blowings-up of \mathbb{P}^2 . From this we can deduce a contradiction.

The proof of rationality when $s \leq 3$ for a general conic bundle also uses the intersection form. A crucial fact is that if an irreducible curve Γ satisfies $\Gamma \cdot \Gamma < 0$, then Γ is unique in its class. If its class is \mathfrak{G} -invariant, Γ itself must be \mathfrak{G} -invariant. We can find a \mathfrak{G} -invariant transcendent basis of $\bar{k}(x, u)$ by using the regular mapping from X to \mathbb{P}^2 or $\mathbb{P}^1 \times \mathbb{P}^1$ induced by such a Γ .

Section 2 is devoted to preliminary discussions from algebraic geometry, Section 3 to a generalized Châtelet surface and Section 4 to a general conic bundle.

2. PRELIMINARIES FROM ALGEBRAIC GEOMETRY.

In this section, we shall state some results in algebraic geometry without proof. For more details, see for instance Hartshorne [Har77], especially Chapter 5 there.

Throughout this section, the ground field k of an algebraic variety is assumed to be algebraically closed.

2.1. Birational mapping. Let X and X' be projective non-singular surfaces, which are mutually birational by $T : X \rightarrow X'$. T can not be defined for finite number of points (which are called fundamental points of T) because the both of numerator and denominator of T becomes zero. T is not injective on finite number of irreducible curves (which are called exceptional curves of T), and T maps every irreducible branch of exceptional curves to a point of X' , which is a fundamental point of T^{-1} .

The complement $O(X)$ of all fundamental points and all exceptional curves is a Zariski open set of X , and T maps $O(X)$ biregularly to $O(X')$ (defined similarly for T^{-1}).

Theorem 2.1 ([Har77, Chapter 5]). *Every birational mapping $T : X \rightarrow X'$ becomes biregular after finite steps of blowing-up at fundamental points of T and T^{-1} respectively (this is valid only for surfaces, and it is not true for higher dimensional varieties).*

A concrete example of such blowings-up is given in the discussion in Subsection 3.1.

2.2. Blowing-up. Let X be a projective non-singular surface, and P be a point on X . Then, there exists uniquely (modulo biregularity) a projective non-singular surface \tilde{X} which satisfies the following (1), (2) and (3). \tilde{X} is called the blowing-up of X at P .

- (1) X and \tilde{X} are mutually birational by $\pi : \tilde{X} \rightarrow X$.
- (2) π is regular and has no fundamental point. π has a unique exceptional curve E_P , which is biregular to the projective line \mathbb{P}^1 , and π maps E_P to P .
- (3) π^{-1} has a unique fundamental point P and has no exceptional curve.

In other words, $X \setminus \{P\}$ and $\tilde{X} \setminus E_P$ are mapped biregularly and π maps E_P to P while π^{-1} is not defined at P .

Roughly speaking, \tilde{X} is the dilation of a point P to a line E_P in \tilde{X} . In the tangent plane of X at P , the direction ratios of tangent vectors correspond to points on E_P . Thus E_P is the set of direction ratios of tangent vectors at P .

2.3. Div(X) and Pic(X). Let X be a projective non-singular surface. The divisor group $\text{Div}(X)$ is defined as the free \mathbb{Z} -module with all irreducible curves on X as basis. Every irreducible curve C on X induces a valuation v_C on the function field $k(X)$, and for $f \in k(X)$, the divisor $\sum v_C(f)C$ is called the principal divisor of f .

When f runs over $k(X)$, the principal divisors form a subgroup of $\text{Div}(X)$, which are called the principal divisor group. It is isomorphic to $k(X)^\times / k^\times$.

The factor group of $\text{Div}(X)$ by the principal divisor group is called the divisor class group or Picard group and denoted by $\text{Pic}(X)$.

Remark 2.1. In more general setting, the definition of Picard group is more complicated, but for a projective non-singular surface, it is nothing but the divisor class group.

Let \tilde{X} be the blowing-up of X at P . Let C be an irreducible curve on X . If C does not pass through P , then $\tilde{C} := \pi^{-1}(C)$ is an irreducible curve on \tilde{X} . If C passes through P , let \tilde{C} be the Zariski closure of $\pi^{-1}(C \setminus \{P\})$ in \tilde{X} , then \tilde{C} is an irreducible curve on \tilde{X} . Besides \tilde{C} , the only one irreducible curve on \tilde{X} is E_P . So identifying C and \tilde{C} , we have $\text{Div}(\tilde{X}) = \text{Div}(X) \oplus \mathbb{Z}$, where \mathbb{Z} represents the free \mathbb{Z} -module with E_P as the base.

Since X and \tilde{X} are birational, the function fields are the same, $k(\tilde{X}) = k(X)$. Taking the factor group by the common principal divisor group, we have $\text{Pic}(\tilde{X}) \simeq \text{Pic}(X) \oplus \mathbb{Z}$, where \mathbb{Z} represents the free \mathbb{Z} -module with E_P as the base.

We shall give the isomorphism more explicitly in the next subsection, using the intersection forms.

2.4. Intersection form.

Theorem 2.2 ([Har77, Chapter 5]). *On $\text{Div}(X) \times \text{Div}(X)$, there exists uniquely a symmetric \mathbb{Z} -bilinear form $D_1 \cdot D_2$ satisfying the following conditions. It is called the intersection form.*

- (1) *If two irreducible curves C_1 and C_2 do not intersect on X , then $C_1 \cdot C_2 = 0$.*
- (2) *If C_1 and C_2 intersects transversally at n points, then $C_1 \cdot C_2 = n$. Here “intersects transversally at P ” means that both C_1 and C_2 are non-singular at P , and tangent vectors of C_1 and C_2 at P are linearly independent.*
- (3) *If D is a principal divisor, then $D \cdot D' = 0$ for all $D' \in \text{Div}(X)$. So that the intersection form is defined on $\text{Pic}(X) \times \text{Pic}(X)$.*

If C_1 and C_2 intersect at n points, but not transversally at some point, then we have $C_1 \cdot C_2 > n$. So, for every two different irreducible curves C_1, C_2 , we have $C_1 \cdot C_2 \geq 0$. But $C \cdot C$ (called the self-intersection number of C) can be < 0 . Note that $C \cdot C$ is determined indirectly using the condition (3).

The relation of the intersection form and blowing-up is as follows.

First, consider $E_P \cdot \tilde{C}$. From (1) and (2) above, we have

- (1') If C does not pass through P , then $E_P \cdot \tilde{C} = 0$.
- (2') If C passes through P , and C is non-singular at P , then $E_P \cdot \tilde{C} = 1$.

(3') Suppose that C passes through P , and C is singular at P . The local equation of C is given by $F(x, y) = 0$ where x and y are local coordinates at P with $x = y = 0$, and $F(x, y)$ is a formal power series of x and y . Since C passes through P , the constant term of F is zero. Since C is singular at P , the coefficients of x and y are also zero. Let ν be the smallest integer of $i + j$ such that the coefficient of $x^i y^j$ is not zero, then $E_P \cdot \tilde{C} = \nu$.

Note that the homogeneous part of degree ν of F induces a polynomial of degree ν in $\frac{y}{x}$, so there are ν roots of $\frac{y}{x}$.

Using this $E_P \cdot \tilde{C}$, we have

$$(2.1) \quad \tilde{C}_1 \cdot \tilde{C}_2 = C_1 \cdot C_2 - (E_P \cdot \tilde{C}_1)(E_P \cdot \tilde{C}_2).$$

For simplicity, suppose that \tilde{C}_1 and \tilde{C}_2 do not intersect on E_P . Since C_1 passes $(E_P \cdot \tilde{C}_1)$ times through P and C_2 passes $(E_P \cdot \tilde{C}_2)$ times through P , there are $(E_P \cdot \tilde{C}_1)(E_P \cdot \tilde{C}_2)$ virtual intersection points on X . This verifies the formula (2.1). More considerations show that the above formula (2.1) is valid even if \tilde{C}_1 and \tilde{C}_2 intersect on E_P . Finally we have

$$(2.2) \quad E_P \cdot E_P = -1,$$

which is obtained using the condition (3) in Theorem 2.2.

Considering the valuation v_C , we see that if D is a principal divisor on X , then $\tilde{D} + (E_P \cdot \tilde{D})E_P$ is a principal divisor on \tilde{X} . This derives the following fact.

Let π^* be a \mathbb{Z} -linear map $\text{Div}(X) \rightarrow \text{Div}(\tilde{X})$ defined by $\pi^*(D) = \tilde{D} + (E_P \cdot \tilde{D})E_P$. Then π^* is injective and maps the principal divisor group to the principal divisor group. So taking the factor group, we get the isomorphism $\text{Pic}(\tilde{X}) \simeq \text{Pic}(X) \oplus \mathbb{Z}$.

Even if $D_1 \equiv D_2$ (\equiv means the identity modulo principal divisor group),

$$(2.3) \quad \tilde{D}_1 \equiv \tilde{D}_2 + \{(E_P \cdot \tilde{D}_2) - (E_P \cdot \tilde{D}_1)\}E_P.$$

2.5. Canonical divisor. Let X be a projective non-singular surface. A canonical divisor of X is defined as follows.

Let $f, g \in k(X)$ be mutually algebraic independent. Let C be an irreducible curve on X and P be a non-singular point of C . Take a local coordinate (x, y) at P

and consider the Jacobian $\frac{\partial(f, g)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix}$, then we can show that $v_C\left(\frac{\partial(f, g)}{\partial(x, y)}\right)$ is

independent of the choice of a point P and the choice of a local coordinate (x, y) .

Canonical divisor of (f, g) is defined as $\sum v_C\left(\frac{\partial(f, g)}{\partial(x, y)}\right)C$.

Take another $f_1, g_1 \in k(X)$ mutually algebraic independent. Then canonical divisor of (f_1, g_1) belongs to the same divisor class with that of (f, g) , namely all canonical divisors determine the unique divisor class in $\text{Pic}(X)$. This is called the canonical divisor class of X and denoted by Ω .

Remark 2.2. In more general setting, the definition of the canonical divisor class is more complicated, but for a projective non-singular surface X , it is nothing but the one defined above.

Example 2.1. For $\mathbb{P}^1 \times \mathbb{P}^1$, we shall determine the intersection form and the canonical divisor.

$\text{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$ has rank 2 as a \mathbb{Z} -module with the basis $x = \infty$ and $u = \infty$ (irreducible curves which do not come from irreducible polynomials in $\bar{k}[x, u]$ are

$x = \infty$ and $u = \infty$). The class of an irreducible curve C of the degree n with respect to x and m with respect to u is $nF + mF'$ where F is the class of $(x = \infty)$ and F' is the class of $(u = \infty)$. For any $c, c' \in \bar{k}$, the representatives of F and F' are chosen as $x = c$ and $u = c'$ respectively.

The intersection form on $\mathbb{P}^1 \times \mathbb{P}^1$ is determined by

$$(2.4) \quad F \cdot F = F' \cdot F' = 0, \quad F \cdot F' = 1.$$

The canonical divisor is

$$(2.5) \quad \Omega = -2F - 2F'.$$

Take $f = x$ and $g = u$, then we have $\frac{\partial(x,u)}{\partial(x,u)} = 1$ since (x, u) is a local coordinate except on the lines $(x = \infty)$ and $(u = \infty)$. In a neighborhood of the line $(x = \infty)$, a local coordinate is (t, u) where $t = \frac{1}{x}$, so $x = \frac{1}{t}$, then $\frac{\partial(x,u)}{\partial(t,u)} = -\frac{1}{t^2}$, thus $v_{(x=\infty)}\left(\frac{\partial(x,u)}{\partial(t,u)}\right) = -2$. The similar result holds for the line $(u = \infty)$. This verifies (2.5). From (2.5) we see that

$$(2.6) \quad C \cdot \Omega = -2(m+n), \quad \Omega \cdot \Omega = 8.$$

Return to a general X and we shall consider the relation with the blowing-up. Let \tilde{X} be the blowing-up of X at a point P . Then the canonical divisor of \tilde{X} is given by

$$(2.7) \quad \Omega_{\tilde{X}} = \pi^* \Omega_X + E_P.$$

This can be derived as follows. Let $f = x$ and $g = y$, where (x, y) is a local coordinate of X at P with $x = y = 0$ at P . Since $\frac{\partial(f,g)}{\partial(x,y)} = 1$, Ω does not pass through P , so $\tilde{\Omega} \cdot E_P = 0$. On the other hand, a local coordinate of \tilde{X} in a neighborhood of E_P is (x, t) where $t = \frac{y}{x}$, so $y = tx$, then $\frac{\partial(x,y)}{\partial(x,t)} = x$, thus $v_{E_P}\left(\frac{\partial(x,y)}{\partial(x,t)}\right) = 1$. This implies $\Omega_{\tilde{X}} = \tilde{\Omega}_X + E_P$.

For other canonical divisors, extending the above relation in the form compatible with the action of π^* , we get (2.7) above.

Since $\pi^* C_1 \cdot \pi^* C_2 = C_1 \cdot C_2$ and $\pi^* C \cdot E_P = 0$ for any irreducible curve C, C_1, C_2 on X , from (2.7) we have

$$(2.8) \quad \begin{aligned} \tilde{C} \cdot \Omega_{\tilde{X}} &= C \cdot \Omega_X + \tilde{C} \cdot E_P, \\ E_P \cdot \Omega_{\tilde{X}} &= -1, \quad \Omega_{\tilde{X}} \cdot \Omega_{\tilde{X}} = \Omega_X \cdot \Omega_X - 1. \end{aligned}$$

2.6. Blowing down. Blowing-down is the inverse operation of the blowing up. Let X be a projective non-singular surface, and assume that there exists an irreducible curve L on X satisfying $L \cdot L = -1$ and $\Omega \cdot L = -1$ (L is necessarily biregular to the projective line \mathbb{P}^1).

Theorem 2.3. *There exists a unique (modulo biregularity) projective non-singular surface \overline{X} such that the blowing-up \tilde{X} at some point $Q \in \overline{X}$ is biregular to X , mapping E_Q to L .*

The surface \overline{X} is called the blowing-down of X by L . Let φ be the biregular mapping $X \rightarrow \tilde{X}$, and π be the projection $\tilde{X} \rightarrow \overline{X}$. For an irreducible curve $C \neq L$ on X , let \overline{C} be the image of C by $\pi \circ \varphi$. Then \overline{C} is an irreducible curve of \overline{X} and all irreducible curves on \overline{X} are obtained in this way. So that identifying C with \overline{C} ,

we get $\text{Div}(X) = \text{Div}(\overline{X}) \oplus \mathbb{Z}$, where \mathbb{Z} represents the free \mathbb{Z} -module with L as the basis.

Let $\overline{\pi}$ be the \mathbb{Z} -linear map from $\text{Div}(X)$ to $\text{Div}(\overline{X})$ defined by $\overline{\pi}(D) = \overline{D - \lambda L}$, where λ is the coefficient of L in D . Then $\overline{\pi}$ is surjective and maps the principal divisor group to the principal divisor group bijectively. The kernel of $\overline{\pi}$ is the free \mathbb{Z} -module with L as the basis. So $\overline{\pi}$ induces the isomorphism $\text{Pic}(X) \simeq \text{Pic}(\overline{X}) \oplus \mathbb{Z}$.

The intersection form on \overline{X} is given by

$$(2.9) \quad \overline{D}_1 \cdot \overline{D}_2 = D_1 \cdot D_2 + (D_1 \cdot L)(D_2 \cdot L).$$

The canonical divisor of \overline{X} is given by

$$(2.10) \quad \Omega_{\overline{X}} = \overline{\pi}(\Omega_X) = \overline{\Omega_X - \lambda L}.$$

We have

$$(2.11) \quad \begin{aligned} \overline{D} \cdot \Omega_{\overline{X}} &= D \cdot \Omega_X - D \cdot L, \\ \Omega_{\overline{X}} \cdot \Omega_{\overline{X}} &= \Omega_X \cdot \Omega_X + 1. \end{aligned}$$

2.7. Blowing-up and down. Let X be a projective non-singular surface and F be an irreducible curve on X satisfying $F \cdot F = 0$ and $F \cdot \Omega = -2$ (F is necessarily biregular to the projective line \mathbb{P}^1). Consider the blowing-up \widetilde{X} at a point P on F . Then we have $\widetilde{F} \cdot \widetilde{F} = -1$ and $\widetilde{F} \cdot \Omega_{\widetilde{X}} = -1$, so that we can consider the blowing-down of \widetilde{X} by \widetilde{F} and obtain $\overline{\widetilde{X}}$.

X and $\overline{\widetilde{X}}$ are birational, but not regular in any direction. Let π_1 be the projection $\widetilde{X} \rightarrow X$ and π_2 be the projection $\overline{\widetilde{X}} \rightarrow \overline{X}$, then $\rho = \pi_2 \circ \varphi \circ \pi_1^{-1}$ is the birational mapping from X to \overline{X} .

The fundamental point of ρ is P , and the exceptional curve of ρ is F . On the other hand, the fundamental point of ρ^{-1} is Q , and the exceptional curve of ρ^{-1} is \overline{E}_P (Q is a point on \overline{E}_P , because $\widetilde{E}_P \cdot E_Q = E_P \cdot \widetilde{F} = 1$).

For an irreducible curve $C \neq F$ on X , \widetilde{C} is an irreducible curve on \widetilde{X} , and besides them, \overline{E}_P is the only irreducible curve on $\overline{\widetilde{X}}$. So that $\text{Div}(X) \simeq \text{Div}(\overline{\widetilde{X}})$, but F is omitted from the basis of $\text{Div}(X)$ and \overline{E}_P is added as the basis of $\text{Div}(\overline{\widetilde{X}})$.

However, we need not replace the basis for Pic . Let $\rho^* = \pi_2^* \circ \pi_1^*$ be the \mathbb{Z} -linear map from $\text{Div}(X)$ to $\text{Div}(\overline{\widetilde{X}})$. The map ρ^* is written as

$$(2.12) \quad \rho^*(D) = \overline{D - \lambda F} + (\widetilde{D} \cdot E_P) \overline{E}_P.$$

The map ρ^* maps $\text{Div}(X)$ to $\text{Div}(\overline{\widetilde{X}})$ bijectively, and maps the principal divisor group to the principal divisor group. So, ρ^* induces an isomorphism of $\text{Pic}(X)$ to $\text{Pic}(\overline{\widetilde{X}})$. Since ρ^* maps F to \overline{E}_P , the divisor class of F is mapped to the divisor class of \overline{E}_P . (More precisely, for a divisor D on X , $D \equiv F$ on X is equivalent with $\rho^*(D) \equiv \overline{E}_P$.)

The intersection form on $\overline{\widetilde{X}}$ is given as follows.

$$(2.13) \quad \begin{aligned} \overline{E}_P \cdot \overline{E}_P &= 0, \quad \widetilde{C} \cdot \overline{E}_P = C \cdot F \text{ for } C \neq F, \\ \widetilde{C}_1 \cdot \widetilde{C}_2 &= C_1 \cdot C_2 + (C_1 \cdot F)(C_2 \cdot F) - (C_1 \cdot F)(\widetilde{C}_2 \cdot E_P) \\ &\quad - (\widetilde{C}_1 \cdot E_P)(C_2 \cdot F). \end{aligned}$$

The canonical divisor of \widetilde{X} is given by

$$(2.14) \quad \Omega_{\widetilde{X}} = \rho^*(\Omega_X) + \overline{E}_P = \widetilde{\Omega_X - \lambda F} + \{(\widetilde{\Omega}_X \cdot E_P) + 1\} \overline{E}_P.$$

Of course we have $\Omega_{\widetilde{X}} \cdot \Omega_{\widetilde{X}} = \Omega_X \cdot \Omega_X$ and $\overline{E}_P \cdot \Omega_{\widetilde{X}} = -2$.

2.8. Iteration of blowings-up and down. Let X be a non-singular projective surface, and P_1, P_2, \dots, P_r be points on X . The successive blowings-up at $\{P_i\}_{1 \leq i \leq r}$ does not depend on the order of the blowings-up (more precisely, the obtained surface by the blowings-up in different orders are mutually biregular).

For successive blowings-up on the once blowing-up E , (namely E_1 is the blowing-up at $P_1 \in X$, E_2 is the blowing-up at $P_2 \in E_1$, E_3 is the blowing-up at $P_3 \in E_2$, and so on) the order of the blowing-up can not be changed. In this case $C \cdot E_i$ is monotonically decreasing.

Let X_1 be the blowing-up of X at a point P_1 , and Y_1 be the blowing-down by some \widetilde{F}_1 , where F_1 is an irreducible curve on X passing through P_1 such that $F_1 \cdot F_1 = 0$ and $F_1 \cdot \Omega = -2$ on X .

The blowing-up of Y_1 at some point Q_1 of Y_1 is biregular with X_1 , mapping E_Q to \widetilde{F}_1 , by the definition of the blowing-down.

Let X_2 be the blowing-up of X_1 at $P_2 \in X_1 \setminus \widetilde{F}_1$. Since $X_1 \setminus \widetilde{F}_1$ is biregular with $Y_1 \setminus \{Q_1\}$, this induces a blowing-up of Y_1 at the corresponding point P'_2 . Blow-down again by some \widetilde{F}_2 , and let Y_2 be the obtained surface.

The blowing-up of Y_2 at some point Q_2 is biregular with the blowing-up of Y_1 at P'_2 . Since X_2 is biregular with the successive blowings-up of Y_1 at Q_1 and P'_2 , we see that X_2 is biregular with the successive blowings-up of Y_2 at Q_2 and Q_1 .

Repeat this r -times. Let X_r be the surface obtained from X by the successive blowings-up at $\{P_i\}$. After each blowing-up, take a suitable blowing-down, and after repeating this r -times, let Y_r be the obtained surface. Then X_r is biregular with the successive blowings-up of Y_r at $\{Q_i\}$. Here we assume that P_i does not lie on F_j for $j < i$ in X_{i-1} (for simplicity, we omit \sim and $-$ for blowing-up and down).

2.9. The surface Y_{rs} . Let X_1 be the blowing up of $\mathbb{P}^1 \times \mathbb{P}^1$ at (a, b) . $\text{Pic}(X_1)$ has rank 3 with basis F, F' and E_1 , where E_1 is the blowing up of the base point (a, b) . The intersection form is the same as $\text{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$ for F and F' and $E_1 \cdot F = E_1 \cdot F' = 0, E_1 \cdot E_1 = -1$ (we take the representative of F as $x = c \neq a$ and the representative of F' as $u = c' \neq b$). The class of an irreducible curve C of the degree n with respect to x and m with respect to u is $nF + mF' - m_1E_1$ where $m_1 = C \cdot E_1$. The canonical divisor is $\Omega = -2F - 2F' + E_1$, so $\Omega \cdot \Omega = 7$.

Let Y be the blowing down of X_1 by $x = a$. $\text{Pic}(Y)$ has rank 2 with the basis F and F' . But the intersection form is different from that of $\text{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$ and $F \cdot F = 0, F \cdot F' = F' \cdot F' = 1$. The class of the above mentioned C is $(n - m_1)F + mF'$. In addition to $x = c (c \neq a)$, E_1 also belongs to the class F . The canonical divisor is $\Omega = -F - 2F'$, so $\Omega \cdot \Omega = 8$. For simplicity, we omit \sim and $-$ for blowing-up and down. The confusion is avoided by seeing C is a curve on which surface.

Starting from Y , consider a similar blowing up and down and let Y_2 be the obtained surface. Repeat this procedure and let Y_r be the surface obtained by r -times blowing up and down. Let Y_{rs} be an s -point blow up of Y_r . $\text{Pic}(Y_{rs})$ has rank $s + 2$ with the basis F, F' and $E_i (1 \leq i \leq s)$. The intersection form is

$F \cdot F = 0$, $F \cdot F' = 1$, $F' \cdot F' = r$, $E_i \cdot F = E_i \cdot F' = E_i \cdot E_j = 0 (i \neq j)$ and $E_i \cdot E_i = -1$. The class of the above mentioned C is $(n - \sum_{j=1}^r m'_j)F + mF' - \sum_{i=1}^s m_i E_i$ with $m_i = C \cdot E_i$ and $m'_j = C \cdot E'_j$ where E'_j is the blowing up used for obtaining Y_r . The canonical divisor is $\Omega = (r - 2)F - 2F' + \sum_{i=1}^s E_i$, so $\Omega \cdot \Omega = 8 - s$.

2.10. Del Pezzo surface.

Definition 2.1. *A non-singular projective surface X is called a del Pezzo surface if it is rational (namely, birational with \mathbb{P}^2 or $\mathbb{P}^1 \times \mathbb{P}^1$ over \bar{k}) and the anti-canonical divisor is ample. The latter condition means that $\Omega \cdot \Omega > 0$ and $\Omega \cdot \Gamma < 0$ for every irreducible curve Γ on X . The degree ω of a del Pezzo surface X is defined to be the self intersection number $\Omega \cdot \Omega$.*

The following is a fundamental theorem for a del Pezzo surface.

Theorem 2.4. *A del Pezzo surface with $\omega \leq 7$ is biregular with $(9 - \omega)$ -point blow up of \mathbb{P}^2 where $\omega = \Omega \cdot \Omega$.*

Proof can be found in Nagata [Nag60a, Nag60b] or Manin [Man86].

A del Pezzo surface with $\omega = 8$ is biregular with $\mathbb{P}^1 \times \mathbb{P}^1$ or one point blow up of \mathbb{P}^2 .

Conversely, a d -point blow up of \mathbb{P}^2 is a del Pezzo surface if and only if $d \leq 8$ and

- (1) any 3 points do not lie on the same line ($d \geq 3$),
- (2) any 6 points do not lie on the same quadratic curve ($d \geq 6$),
- (3) there exists no cubic curve which passes through all 8 points and singular at one of them ($d = 8$).

Theorem 2.5. *On a del Pezzo surface S , consider the following condition (2.15) on a class $F \in \text{Pic}(S)$:*

$$(2.15) \quad F \cdot F = 0, F \cdot \Omega = -2 \text{ and } F \text{ contains an irreducible curve.}$$

Then we have

- (1) *For any point $P \in S$, there exists a unique curve $C \in F$, which passes through P (C is irreducible except finite number of them),*
- (2) *$-\Omega - F$ (resp. $-2\Omega - F$, resp. $-4\Omega - F$) also satisfies the condition (2.15) for $\omega = 4$ (resp. $\omega = 2$, resp. $\omega = 1$).*

Theorem 2.6. *On a del Pezzo surface S , consider the following condition (2.16) on a class $\Gamma \in \text{Pic}(S)$:*

$$(2.16) \quad \Gamma \cdot \Gamma = -1, \Gamma \cdot \Omega = -1 \text{ and } \Gamma \text{ contains an irreducible curve.}$$

Obviously the irreducible curve is unique in its class, so denote it by the same symbol Γ . Then $-\Omega - \Gamma$ (resp. $-2\Omega - \Gamma$) also satisfies the condition (2.16) for $\omega = 2$ (resp. $\omega = 1$).

For a d -point blow-up of \mathbb{P}^2 , we can write down explicitly all the classes which satisfy (2.15) or (2.16), and check the validity of Theorem 2.5 and Theorem 2.6. Theorem 2.5 and Theorem 2.6 are valid also for a general del Pezzo surface by Theorem 2.4.

3. GENERALIZED CHÂTELET SURFACE

We shall examine the rationality of $K = k(x, y, z)$ as in (1.2) $z^2 = ay^2 + P(x)$ under the assumptions (C1), ..., (C5) in Subsection 1.1.

Let l be the splitting field of $P(x)$. Then $\sqrt{a} \in l$ because of the condition (C3). Thus $l(x, y, z)$ is l -rational and $l(x, y, z) = l(x, u)$ where $u = z + \sqrt{a}y$. The field l is a Galois extension of k , and we write $\mathfrak{G} = \text{Gal}(l/k)$ and $N = \text{Gal}(l/k(\sqrt{a}))$. The group \mathfrak{G} acts on x trivially, and N acts on u trivially; for any $\sigma \in \mathfrak{G} \setminus N$, $\sigma : u \mapsto z - \sqrt{a}y = \frac{P(x)}{u}$.

The automorphism $T : (x, u) \mapsto (x, \frac{P(x)}{u})$ of $l(x, u)$ induces an \bar{l} -birational transformation of $\mathbb{P}^1 \times \mathbb{P}^1$. After successive blowings-up and blowings-down, we obtain a surface X defined over l , on which T acts as a biregular automorphism.

3.1. Biregularization of T . Let T be the birational transformation of $\mathbb{P}^1 \times \mathbb{P}^1$ defined by $T : x \mapsto x, u \mapsto \frac{P(x)}{u}$. Let $r = \deg P$ and c_1, c_2, \dots, c_r be the roots of P . T has $r + 1$ fundamental points and $r + 1$ exceptional curves. Fundamental points are $P_i : x = c_i, u = 0$ ($1 \leq i \leq r$) and $P_{r+1} : x = u = \infty$. Exceptional curves are $x = c_i$ ($1 \leq i \leq r$) and $x = \infty$.

Consider the blowings-up for each P_i . Let X_1 be the blowing-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at P_1 and T_1 be the lifting of T to X_1 . X_1 is a surface in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ defined by $\frac{u}{x-c_1} = t_1$. E_1 is the curve $x = c_1, u = 0$, and $\widetilde{x=c_1}$ is the curve $x = c_1, t_1 = \infty$.

We write $P(x) = b \prod_{j=1}^r (x - c_j)$. By T_1 , each point $(c_1, u, \infty) \in \widetilde{x=c_1}$ is mapped to $(c_1, 0, \frac{b \prod_{j \neq 1} (c_1 - c_j)}{u}) \in E_1$, and each point $(c_1, 0, t_1) \in E_1$ is mapped to $(c_1, \frac{b \prod_{j \neq 1} (c_1 - c_j)}{t_1}, \infty) \in \widetilde{x=c_1}$. So T_1 maps E_1 biregularly to $\widetilde{x=c_1}$.

Next let X_2 be the blowing-up of X_1 at P_2 and T_2 be the lifting of T_1 to X_2 . Repeat this r times so that T_r maps E_i biregularly to $\widetilde{x=c_i}$ for $1 \leq i \leq r$:

$$\begin{array}{ccc}
 X_r & \xrightarrow{T_r} & X_r \\
 \downarrow & & \downarrow \\
 \vdots & & \vdots \\
 \downarrow & & \downarrow \\
 X_2 & \xrightarrow{T_2} & X_2 \\
 \downarrow & & \downarrow \\
 X_1 & \xrightarrow{T_1} & X_1 \\
 \downarrow & & \downarrow \\
 \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{T} & \mathbb{P}^1 \times \mathbb{P}^1.
 \end{array}$$

Now the only fundamental point of T_r is $P_{r+1} : x = u = \infty$, and the only exceptional curve of T_r is $x = \infty$.

The blowing-up at P_{r+1} does not make T_{r+1} biregular. Let X_{r+1} be the blowing-up of X_r at P_{r+1} and T_{r+1} be the lifting of T_r to X_{r+1} . X_{r+1} is a surface in $X_r \times \mathbb{P}^1$ defined by $\frac{u}{x} = t_{r+1}$. E_{r+1} is the curve $x = \infty, u = \infty$ and $\widetilde{x=\infty}$ is the curve $x = \infty, t_{r+1} = 0$.

T_{r+1} maps $\widetilde{x=\infty}$ to one point $P_{r+2} \in E_{r+1}$ defined by $t_{r+1} = \infty$, and maps E_{r+1} to P_{r+2} . So exceptional curves of T_{r+1} are $\widetilde{x=\infty}$ and E_{r+1} while the only fundamental point is P_{r+2} .

So, blow up again. Let X_{r+2} be the blowing-up of X_{r+1} at P_{r+2} and T_{r+2} be the lifting of T_{r+1} to $\widehat{X_{r+2}}$. X_{r+2} is a surface in $X_{r+1} \times \mathbb{P}^1$ defined by $\frac{t_{r+1}}{x} = t_{r+2} = \frac{u}{x^2}$. Then T_{r+2} maps $x = \infty$ to one point $P_{r+3} \in E_{r+2}$ defined by $t_{r+2} = \infty$. If $r = 3$, then T_5 maps E_4 biregularly to E_5 , but if $r > 3$, then T_{r+2} maps both of E_{r+1} and E_{r+2} to one point P_{r+3} .

Repeating this r times. Let X be the obtained surface and T_{2r} be the lifting of T to X . Then T_{2r} becomes biregular, namely T_{2r} maps $x = \infty$ to E_{2r} , and E_{r+i} to E_{2r-i} ($1 \leq i \leq r-1$) biregularly:

$$\begin{array}{ccc}
 X & \xrightarrow{T_{2r}} & X \\
 \downarrow & & \downarrow \\
 \vdots & & \vdots \\
 \downarrow & & \downarrow \\
 X_{r+2} & \xrightarrow{T_{r+2}} & X_{r+2} \\
 \downarrow & & \downarrow \\
 X_{r+1} & \xrightarrow{T_{r+1}} & X_{r+1} \\
 \downarrow & & \downarrow \\
 X_r & \xrightarrow{T_r} & X_r.
 \end{array}$$

Thus, T becomes biregular after $2r$ blowings-up in total, once for each $1 \leq i \leq r$, and r times for P_{r+1} . We denote the obtained surface by X .

3.2. Reduction to the even degree case. Without loss of generality, we can assume that $\deg P = r$ is even, by the following reason.

Suppose that $\deg P = r$ is odd, and put $r = 2s - 1$. Put $x' = \frac{1}{x}$, $y' = x'^s y$, $z' = x'^s z$, then $z^2 = ay^2 + P(x)$ is re-written as $z'^2 = ay'^2 + x'^{2s} P(\frac{1}{x'})$. When $P(x) = \sum_{i=0}^{2s-1} a_i x^i$ with $a_{2s-1} \neq 0$, $P_1(x) := x'^{2s} P(\frac{1}{x'}) = \sum_{i=0}^{2s-1} a_i x'^{2s-i}$ is a polynomial with the degree $2s$. Since $k(x, y, z) = k(x', y', z')$, the k -rationality problem of $k(x, y, z)$ is reduced to that of $k(x', y', z')$ for the polynomial $P_1(x)$ of even degree.

Since the root of $P_1(x)$ are 0 and $\{\frac{1}{c_i}\}_{1 \leq i \leq r}$, where $\{c_i\}$ are the roots of $P(x)$, the conditions (C1), ..., (C5) are satisfied for $P_1(x)$ also (note that we can assume $P(0) \neq 0$ without loss of generality). When k is a finite field and $|k|$ is small, we may take a finite extension $k' \supset k$ which satisfies the conditions (C1), ..., (C5), and continue the argument above. Note that, if $k(x, y, z)$ is k -rational, then $k'(x, y, z)$ is k' -rational.

3.3. Another biregularization of T . In this subsection, after reaching r blowings-up (this surface is X_r in the subsection 3.1), we shall proceed in another way. Blow up X_r at the point $P_{r+1} : x = \infty, u = \infty$ (this surface is $\widehat{X_{r+1}}$, which is a surface in $X_r \times \mathbb{P}^1$ defined by $\frac{u}{x} = t_{r+1}$), and then blow-down it by $\widehat{x = \infty}$. We denote the obtained surface by Y_1 . We lift up and lift down $T : X_r \rightarrow X_r$ to get $T : Y_1 \rightarrow Y_1$. T maps $E_{r+1} : x = \infty, u = \infty$ to one point $P_{r+2} \in E_{r+1}$ defined by $t_{r+1} = \infty$. So the only fundamental point of T is P_{r+2} , and the only exceptional curve is E_{r+1} . In $\text{Div}(Y_1)$, $(x = \infty)$ disappears and is replaced by E_{r+1} .

Blow up Y_1 at P_{r+2} (this surface is in $Y_1 \times \mathbb{P}^1$ defined by $t_{r+2} = \frac{u}{t_{r+1}} = \frac{u}{x^2}$), and then blow down by E_{r+1} . In the obtained surface Y_2 , E_{r+1} disappears and is replaced by $E_{r+2} : x = \infty, t_1 = \infty$. The only fundamental point of T is $P_{r+3} \in E_{r+2}$ defined by $t_{r+2} = \infty$, and the only exceptional curve is E_{r+2} .

For an even r , repeat this process $\frac{r}{2}$ times. On the surface $Y_{\frac{r}{2}}$, T becomes biregular, and maps $E_{\frac{3r}{2}}$ biregularly to $E_{\frac{3r}{2}}$, as studied in Subsection 3.1. All E_{r+j} ($j < \frac{r}{2}$) disappear by the blowings-down. We shall denote the obtained $Y_{\frac{r}{2}}$ by Y .

Thus, for an even r , T becomes biregular after r blowings-up and $\frac{r}{2}$ blowings-up and down.

3.4. Pic(Y) as a Galois module. Let k be an algebraically non-closed field, and K be an algebraic function field with two variables over k . Namely, K is a finite extension of the rational function field with two variables over k such that k is algebraically closed in K .

Let \bar{k} be a fixed algebraic closure of k . The k -automorphism group of \bar{k} is isomorphic to $\text{Gal}(k^{\text{sep}}/k)$, where k^{sep} is the separable closure of k , because every k -automorphism of k^{sep} is extended uniquely to \bar{k} . $G := \text{Gal}(k^{\text{sep}}/k)$ acts on $\bar{k} \otimes_k K$, assuming that it acts on K trivially, namely $G \ni \sigma \mapsto \bar{\sigma} = \sigma \otimes \text{id}_K$.

Assume that $\bar{k} \otimes_k K$ is \bar{k} -rational, namely $\bar{k} \otimes_k K = \bar{k}(u, v)$ for some u, v . Let u^σ, v^σ be the image of u, v by the action of $\bar{\sigma}$, then we have $\bar{k}(u, v) = \bar{k}(u^\sigma, v^\sigma)$, so that $u \mapsto u^\sigma, v \mapsto v^\sigma$ induces a \bar{k} -automorphism T_σ of $\bar{k}(u, v)$. T_σ is different from $\bar{\sigma}$, because T_σ acts trivially on \bar{k} . Let $\tilde{\sigma}$ be a \bar{k} -automorphism of $\bar{k}(u, v)$ such that $\tilde{\sigma}$ acts naturally on \bar{k} , and acts trivially on u and v . Then we have $\bar{\sigma} = T_\sigma \circ \tilde{\sigma}$.

T_σ induces a birational transformation of $\mathbb{P}^1 \times \mathbb{P}^1$, while $\tilde{\sigma}$ induces a homeomorphic transformation in Zariski topology of $\mathbb{P}^1 \times \mathbb{P}^1$. Suppose that after suitable blowings-up or blowings-up and down of $\mathbb{P}^1 \times \mathbb{P}^1$, all of T_σ become biregular on the obtained surface Y . The lifting of $\tilde{\sigma}$ to Y is homeomorphic in Zariski topology. So the action of $\bar{\sigma}$ induces a permutation of irreducible curves, and $\text{Div}(Y)$ becomes a permutation G -module.

Since the action of $\bar{\sigma}$ keeps the function field $\bar{k} \otimes_k K = \bar{k}(u, v)$ invariant, it keeps the principal divisor group invariant, so taking the factor module, we see that $\text{Pic}(Y)$ is also a G -module.

But since $\text{Pic}(Y)$ is of finite rank as a \mathbb{Z} -module, and since $u, v \in \bar{k}(x, y)$ actually belongs to $l(x, y)$ for some finite extension of k , $\text{Pic}(Y)$ is a \mathfrak{G} -module, where $\mathfrak{G} = \text{Gal}(l/k)$, l being a sufficiently large finite Galois extension of k .

Thus, $\text{Pic}(Y)$ becomes a \mathfrak{G} -lattice. Here a \mathfrak{G} -lattice means a free \mathbb{Z} -module of finite rank with the action of \mathfrak{G} as automorphisms.

3.5. Manin's criterion. Let K' be another algebraic function field with two variables over k such that $\bar{k} \otimes_k K'$ is \bar{k} -rational. Let $\bar{k} \otimes_k K' = \bar{k}(u', v')$. $G = \text{Gal}(k^{\text{sep}}/k)$ acts on $\bar{k} \otimes_k K'$ as $G \ni \sigma \mapsto \bar{\sigma}' = \sigma \otimes \text{id}_{K'}$.

By the discussions in the previous subsection, $\bar{\sigma}'$ can be written as $\bar{\sigma}' = T'_\sigma \circ \tilde{\sigma}'$, where T'_σ is a \bar{k} -automorphism of $\bar{k}(u', v')$ and $\tilde{\sigma}'$ is a \bar{k} -automorphism of $\bar{k}(u', v')$ which acts on \bar{k} naturally and acts on u' and v' trivially. T'_σ induces a birational transformation of $\mathbb{P}^1 \times \mathbb{P}^1$. Suppose that after finite steps of blowings-up (and down), all T'_σ becomes biregular on the obtained surface Y' , so we can regard $\text{Pic}(Y')$ as a \mathfrak{G} -lattice, where $\mathfrak{G} = \text{Gal}(l/k)$ for sufficiently large finite Galois extension l of k .

Proposition 3.1. *K is k -isomorphic to K' if and only if there exists a \bar{k} -isomorphism T from $\bar{k} \otimes_k K$ to $\bar{k} \otimes_k K'$ which commutes with the action of G , namely for $\forall \sigma \ni G = \text{Gal}(k^{\text{sep}}/k), T \circ \bar{\sigma} = \bar{\sigma}' \circ T$.*

Proof. Suppose that K is k -isomorphic to K' and let T_0 be the k -isomorphism. Then T_0 is naturally extended to a \bar{k} -isomorphism T from $\bar{k} \otimes_k K$ to $\bar{k} \otimes_k K'$, $T = \text{id}_{\bar{k}} \otimes T_0$. Evidently T commutes with the action of G .

Conversely, suppose that a required \bar{k} -isomorphism T exists. Since T commutes with the action of G , T and T^{-1} map the fixed field of G to each other. However, the fixed field of $\bar{k} \otimes_k K$ (resp. $\bar{k} \otimes_k K'$) of G is K (resp. K'), and the restriction of T on K becomes a k -isomorphism from K to K' . \square

Definition 3.1. Let H be a finite group and M be an H -lattice (i.e. a free \mathbb{Z} -module of finite rank with the action of H as automorphisms). An H -lattice M is called permutation if M has a \mathbb{Z} -basis permuted by H , i.e. $M \simeq \bigoplus_{1 \leq i \leq m} \mathbb{Z}[H/H_i]$ for some subgroups H_1, \dots, H_m of H . We say that two H -lattices M_1 and M_2 are similar if there exist permutation H -lattices P_1 and P_2 such that $M_1 \oplus P_1 \simeq M_2 \oplus P_2$.

Proposition 3.2 (Manin [Man67]). Let $\text{Pic}(Y)$ (resp. $\text{Pic}(Y')$) be the \mathfrak{S} -lattice corresponding to K (resp. K') as in Subsection 3.4. If K is k -isomorphic to K' , then $\text{Pic}(Y)$ and $\text{Pic}(Y')$ are similar, i.e. there exist permutation \mathfrak{S} -lattices P_1 and P_2 such that $\text{Pic}(Y) \oplus P_1 \simeq \text{Pic}(Y') \oplus P_2$.

Proof. Assume the existence of a required \bar{k} -isomorphism T from $\bar{k}(u, v)$ to $\bar{k}(u', v')$. Then T induces a birational transformation of $\mathbb{P}^1 \times \mathbb{P}^1$. After suitable blowings-up or blowings-up and down, T is lifted to a birational map from Y to Y' . Though it may not be biregular on Y , after further suitable blowings-up, we can reach the surfaces Z and Z' , on which T (and T^{-1}) becomes biregular.

Since T is biregular, we have $\text{Pic}(Z) \simeq \text{Pic}(Z')$ as \mathbb{Z} -modules. Since T commutes with the action of G , (then their liftings commutes also), $\text{Pic}(Z) \simeq \text{Pic}(Z')$ as \mathfrak{S} -lattices also.

Only remained to prove is that $\text{Pic}(Z) \simeq \text{Pic}(Y) \oplus P$ for some permutation \mathfrak{S} -lattice P .

Let $\{E_j\}$ be the successive blowings-up to reach Z from Y . Since T commutes with the action of G , the set of fundamental points of T is G -invariant, and the action of G induces permutations of $\{E_j\}$.

Let $\{e_i\}$ be the basis of $\text{Pic}(Y)$ as a free \mathbb{Z} -module. Then $\text{Pic}(Z)$ is a free \mathbb{Z} -module with the basis $\{\pi^*e_i\} \cup \{E_j\}$, where π^* is a \mathbb{Z} -linear map from $\text{Pic}(Y)$ to $\text{Pic}(Z)$, obtained by the iteration of π^* mentioned at the end of Subsection 2.4. Let M_1 (resp. M_2) be a free \mathbb{Z} -module with the basis $\{\pi^*e_i\}$ (resp. $\{E_j\}$). Then $\text{Pic}(Z) \simeq M_1 \oplus M_2$ as \mathbb{Z} -modules. However, M_2 is a permutation \mathfrak{S} -lattice as mentioned above. We can show that M_1 is also a \mathfrak{S} -lattice which is isomorphic to $\text{Pic}(Y)$. \square

Corollary 3.1. If K is k -isomorphic to K' , then $H^1(\mathfrak{S}, \text{Pic}(Y)) \simeq H^1(\mathfrak{S}, \text{Pic}(Y'))$ and $\widehat{H}^{-1}(\mathfrak{S}, \text{Pic}(Y)) \simeq \widehat{H}^{-1}(\mathfrak{S}, \text{Pic}(Y'))$, where H^1 is Galois cohomology and \widehat{H}^{-1} is Tate cohomology.

This comes from $H^1(\mathfrak{S}, P) = \widehat{H}^{-1}(\mathfrak{S}, P) = 0$ for a permutation \mathfrak{S} -lattice P .

Especially, if K is k -rational, then $H^1(\mathfrak{S}, \text{Pic}(Y)) = \widehat{H}^{-1}(\mathfrak{S}, \text{Pic}(Y)) = 0$ by the following reason.

The k -rationality of K means that K is k -isomorphic to the two dimensional rational function field $K' = k(x, y)$. In this case, $\bar{k} \otimes_k K' = \bar{k}(x, y)$ and $\bar{\sigma}'$ acts trivially on x and y . So, $Y' = \mathbb{P}^1 \times \mathbb{P}^1$ and $\text{Pic}(Y')$ is a trivial \mathfrak{S} -lattice, so that

$H^1(\mathfrak{G}, \text{Pic}(Y')) = \widehat{H}^{-1}(\mathfrak{G}, \text{Pic}(Y')) = 0$. In other words, $H^1(\mathfrak{G}, \text{Pic}(Y)) \neq 0$ or $\widehat{H}^{-1}(\mathfrak{G}, \text{Pic}(Y)) \neq 0$ is a criterion for the k -irrationality of K .

3.6. Proof of Theorem 1.4. As mentioned in Subsection 1.3, Theorem 1.4 was proved already by Colliot-Thélène and Sansuc [San81, Proposition 1 (v)] (see also [CTS94]). The following proof is included for the convenience of the reader.

Let K be the quadratic extension of $k(x, y)$ defined by (1.2): $z^2 = ay^2 + P(x)$ with conditions (C1), ..., (C5) in Subsection 1.1. Then $\bar{k} \otimes_k K = \bar{k}(x, u)$ where $u = z + \sqrt{a}y$. For $\sigma \in G = \text{Gal}(k^{\text{sep}}/k)$, T_σ is either the identity or equal to $T : x \mapsto x, u \mapsto \frac{P(x)}{u}$ according to whether \sqrt{a} is invariant by σ or not. T induces a birational transformation of $\mathbb{P}^1 \times \mathbb{P}^1$, and it becomes biregular on the obtained surface Y , mentioned in Subsection 3.3. Then $\text{Pic}(Y)$ is a free \mathbb{Z} -module of rank $r + 2$ with the basis $E_i (1 \leq i \leq r)$, F and F' . (We assume that r is even.)

We shall determine the structure of $\text{Pic}(Y)$ as a \mathfrak{G} -lattice, where $\mathfrak{G} = \text{Gal}(l/k)$, l being the splitting field of $P(x)$ over k .

As studied in Subsection 3.3, T maps E_i to $(\widetilde{x=c_i})$ for $1 \leq i \leq r$, and $(\widetilde{u=c})$ to $(u = \frac{P(x)}{c})$. The next question is what divisor classes they belong to.

Let π^* and ρ^* be a \mathbb{Z} -linear map from $\text{Div}(\mathbb{P}^1 \times \mathbb{P}^1)$ to $\text{Div}(Y)$, obtained by the iteration of π^* mentioned at the end of Subsection 2.4 and Subsection 2.7. Then we have

$$\begin{aligned} \pi^*(x=c_i) &= (\widetilde{x=c_i}) + E_i \text{ for } 1 \leq i \leq r, \\ \rho^*(u = \frac{P(c)}{c}) &= (\widetilde{u = \frac{P(c)}{c}}) + \sum_{j=1}^r E_j + \frac{r}{2}F. \end{aligned} \quad (3.1)$$

So that in $\text{Pic}(Y)$, we have

$$\begin{aligned} (\widetilde{x=c_i}) &\equiv F - E_i, \\ (\widetilde{u = \frac{P(x)}{c}}) &\equiv F' + \frac{r}{2}F - \sum_{j=1}^r E_j. \end{aligned} \quad (3.2)$$

Therefore, the action of $\bar{\sigma} = T_\sigma \circ \tilde{\sigma}$ on $\text{Pic}(Y)$ is represented by the following matrix g_σ , with $E_1, E_2, \dots, E_r, F, F'$ as the basis in this order.

$$\begin{aligned} (3.3) \quad \text{For } \sigma \in N = \text{Gal}(l/k(\sqrt{a})), \quad g_\sigma &= \begin{pmatrix} A_\sigma & 0 \\ 0 & I_2 \end{pmatrix}, \\ \text{for } \sigma \in \mathfrak{G} \setminus N, \quad g_\sigma &= \begin{pmatrix} -A_\sigma & 1 & 0 \\ 0 & 1 & 0 \\ -1 & \frac{r}{2} & 1 \end{pmatrix}, \end{aligned}$$

where 1 (resp. 0, -1) stands for the matrix whose entries are all 1 (resp. 0, -1).

A_σ is the permutation matrix of the permutation of $\{c_i\}$ induced by σ . Suppose that $P(x)$ is a product of r' irreducible polynomials. Then, the set of roots $\{c_i\}$ of $P(x)$ is divided into r' blocks, each of which consists of the roots of the same irreducible component. Each block is a transitive part by the action of \mathfrak{G} . Since each irreducible component is assumed to be irreducible also over $k(\sqrt{a})$, the action of N is also transitive on each block. The block is called even (resp. odd), when the degree of the corresponding irreducible polynomial is even (resp. odd).

Let M_0 be the submodule spanned by $\{E_i | 1 \leq i \leq r\}$. An element of M_0 is written as $\sum_{i=1}^r a_i E_i$, $a_i \in \mathbb{Z}$. Let s_j be the sum of a_i when i runs over the j -th

block. Let M_e be the submodule of M_0 , consisting of elements such that $\sum_{j=1}^{r'} s_j$ is even. Let M_b be the submodule of M_0 , consisting of elements such that s_j is even for every j . We have $M_0/M_e \simeq \mathbb{Z}/2\mathbb{Z}$, $M_0/M_b \simeq (\mathbb{Z}/2\mathbb{Z})^{r'}$ and $M_e/M_b \simeq (\mathbb{Z}/2\mathbb{Z})^{r'-1}$, where r' is the number of the blocks.

By definition, we have $\hat{H}^{-1}(\mathfrak{G}, \text{Pic}(Y)) = Z/B$, where Z and B are submodules of $\text{Pic}(Y)$ defined by

$$(3.4) \quad Z = \text{Ker}\left(\sum_{\sigma} g_{\sigma}\right), \quad B \text{ is the module spanned by } \bigcup_{\sigma} \text{Im}(\sigma - \text{id}),$$

where the summation and the union are taken over $\sigma \in \mathfrak{G}$.

The sum $\sum g_{\sigma}$ is zero except the last row and the $(r+1)$ -th column, so that the rank of Z is r and the projection to M_0 (projection as \mathbb{Z} -modules) is injective. Let Z' and B' be the images of the projection of Z and B respectively, then $Z/B \simeq Z'/B'$.

We see that $Z' = M_e$ and $B' = M_b + (\sum_{i=1}^r E_i)\mathbb{Z}$. Since $M_e/M_b \simeq (\mathbb{Z}/2\mathbb{Z})^{r'-1}$, and since $\sum_{i=1}^r E_i \in M_b$ if and only if odd block does not exist, we have

$$(3.5) \quad \hat{H}^{-1}(\mathfrak{G}, \text{Pic}(Y)) \simeq \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{r'-1} & \text{if odd block does not exist,} \\ (\mathbb{Z}/2\mathbb{Z})^{r'-2} & \text{if odd blocks exist.} \end{cases}$$

As for $H^1(\mathfrak{G}, \text{Pic}(Y))$, we proceed as follows. In general for a \mathfrak{G} -lattice M , $H^1(\mathfrak{G}, M)$ is isomorphic to \hat{H}^{-1} of the dual lattice M' . So that as for $H^1(\mathfrak{G}, \text{Pic}(Y))$, it suffices to calculate \hat{H}^{-1} for the transposed matrix of (3.3). The calculation shows that $H^1(\mathfrak{G}, \text{Pic}(Y)) \simeq \hat{H}^{-1}(\mathfrak{G}, \text{Pic}(Y))$, though the matrix (3.3) is not symmetric.

Thus, the proof of Theorem 1.4 has been completed. Note that when $\deg P$ is odd, $\hat{H}^{-1}(\mathfrak{G}, \text{Pic}(Y)) \simeq (\mathbb{Z}/2\mathbb{Z})^{r'-1}$, because the reduction in Subsection 3.2 implies that r' increases by 1 and odd block obviously exist, so we have $(r'+1)-2 = r'-1$.

3.7. \mathfrak{G} -invariant classes. From the matrices (3.3), we have the followings.

Proposition 3.3. *The submodule $\text{Pic}(Y)^{\mathfrak{G}}$ of \mathfrak{G} -invariant classes has rank 2 with the basis F and Ω . We have $F \cdot F = 0$, $F \cdot \Omega = -2$ and $\Omega \cdot \Omega = 8 - r$.*

Proof. From (3.3), $\sum_{\sigma} g_{\sigma}$ is represented by

$$(3.6) \quad \sum_{\sigma} g_{\sigma} = \frac{|\mathfrak{G}|}{2} \begin{pmatrix} O_r & 1 & 0 \\ 0 & 2 & 0 \\ -1 & \frac{r}{2} & 2 \end{pmatrix}$$

with the basis E_i and F, F' , where O_r is the $r \times r$ matrix whose entries are all zero.

Since the image of $\sum_{\sigma} g_{\sigma}$ is contained in $\text{Pic}(Y)^{\mathfrak{G}}$ with finite index, we see that $\text{Pic}(Y)^{\mathfrak{G}}$ is generated by F and $2F' + \frac{r}{2}F - \sum_{i=1}^r E_i$. But since $\Omega_Y = -2F' + (\frac{r}{2} - 2)F + \sum_{i=1}^r E_i$ as stated in Subsection 2.9, $\text{Pic}(Y)^{\mathfrak{G}}$ is generated by F and Ω .

$F \cdot F = 0$ etc. is easily obtained by the intersection form of Y_{rs} stated before. \square

Note that if a curve C is \mathfrak{G} -invariant, then its class is also \mathfrak{G} -invariant. The converse is not true, because C can be moved in the same class.

3.8. Proof of Theorem 1.5. If $k(x, y, z)$ is k -rational, write $k(x, y, z) = k(t, s)$ for some t and s . Thus $\bar{l}(x, u) = \bar{l}(t, s)$ with \mathfrak{G} -invariant t, s .

Let Y be the algebraic surface obtained in Subsection 3.3. It follows that Y is k -birational with $\mathbb{P}^1 \times \mathbb{P}^1$. Here “ k -birational” means that there exists a birational mapping $\mathbb{P}^1 \times \mathbb{P}^1 \rightarrow Y$ which commutes with the action of \mathfrak{G} , where \mathfrak{G} acts trivially on t and s .

Let Φ be a k -birational mapping $\mathbb{P}^1 \times \mathbb{P}^1 \rightarrow Y$. After finite steps of blowings-up of $\mathbb{P}^1 \times \mathbb{P}^1$ and Y respectively, Φ is lifted to a biregular mapping $Z \rightarrow Z'$.

For $a, b \in k$, the lines $t = a$ and $s = b$ are \mathfrak{G} -invariant in $\mathbb{P}^1 \times \mathbb{P}^1$, so that their images are also \mathfrak{G} -invariant in Y or in Z' . Suppose that $t = a$ is not an exceptional curve of Φ and does not pass through a fundamental point of Φ , then the values of intersection form $(t = a) \cdot (t = a) = 0, (t = a) \cdot \Omega = -2$ are kept invariant under the blowings-up, so the image C in Z' also satisfies $C \cdot C = 0$ and $C \cdot \Omega = -2$ in Z' . Z' is obtained from Y by successive blowings-up $\{E'_j\}$. By each blowing-up, $C \cdot C$ is decreased by $(C \cdot E'_j)^2$ and $\Omega \cdot C$ is increased by $C \cdot E'_j$, by (2.1) and (2.8).

Thus we have on the surface Y

$$(3.7) \quad \begin{aligned} C \cdot C &= \sum_j m_j^2, \\ \Omega \cdot C &= -2 - \sum_j m_j, \end{aligned}$$

where $m_j = C \cdot E'_j$. We represent the class of C with $\nu F - m\Omega$, then we have

$$(3.8) \quad \begin{aligned} C \cdot C &= 4m\nu + \omega m^2, \\ \Omega \cdot C &= -2\nu - m\omega \text{ where } \omega = \Omega \cdot \Omega. \end{aligned}$$

Combining (3.7) with (3.8), we get

$$(3.9) \quad \begin{aligned} \sum_j m_j^2 &= 4m\nu + \omega m^2, \\ \sum_j m_j &= 2\nu + m\omega - 2. \end{aligned}$$

From $C \cdot F = 2m$, we have $m \geq 0$, and $m = 0$ means that C is $(x = c)$ for some $c \in k$. For $m > 0$, we have $0 \leq m_j \leq 2m$ by the following reason.

Let $(x = c_j)$ be the line passing through the base point of E'_j . $\widetilde{C} \cdot (\widetilde{x = c_j}) \geq 0$ on the blowing-up surface implies $C \cdot E'_j \leq C \cdot (x = c_j) = C \cdot F = 2m$. Note that for successive blowings-up $\{E'_j\}$ (namely E'_1 is the blowing-up at $P_1 \in Y$, E'_2 is the blowing-up at $P_2 \in E'_1$, E'_3 is the blowing-up at $P_3 \in E'_2$, and so on), $C \cdot E'_j$ is monotone decreasing. Any successive blowing-up also satisfies $C \cdot E'_j \leq 2m$.

Thus we have $0 \leq \sum_j m_j^2 \leq 2m \sum_j m_j$, so that

$$(3.10) \quad 4m\nu + \omega m^2 \leq 2m(2\nu + m\omega - 2)$$

which yields $\omega m^2 \geq 4m$. This is impossible if $\omega \leq 0$ and $m > 0$. Since $\omega = 8 - r$, if $r \geq 8$, then any \mathfrak{G} -invariant curve other than $x = \text{const.}$ cannot become the image of $t = a$.

The same holds for $s = b$. Since t and s are algebraically independent, at least one of t and s depends on u so that $m \geq 1$. Thus we reach a contradiction. \square

3.9. Reduction to a del Pezzo surface. Since we are assuming that r is even, the remained cases are $r = 6$ and $r = 4$. We shall continue the discussion for these cases.

Let P'_1 be a point on Y such that $m_1 > m$. The point P'_1 lies on $(x = c)$ for some c . Note that $c \neq c_i$, since $C \cdot E_i = C \cdot \Omega = m$, so that for any point P on $(x = c_i)$, $C \cdot E_P > m$ can not occur.

Let Y_1 be the blowing-up of Y at P'_1 , and Y'_1 be the blowing-down of Y_1 by $x = c$. Y_1 is biregular with the blowings-up of Y'_1 at some Q_1 , where

$$(3.11) \quad C \cdot E_{Q_1} = C \cdot F - C \cdot E'_1 = 2m - m_1 < m.$$

Let P'_2 be a point on Y_1 such that $m_2 > m$. The point P'_2 does not lie on $(x = c)$, because of $C \cdot E'_1 > m$.

Since $Y_1 \setminus (x = c)$ is biregular with $Y'_1 \setminus \{Q_1\}$, the blowing-up at P'_2 induces the blowing-up of Y'_1 at the corresponding point P''_2 . After the blowing-up at P''_2 , blow-down by $(x = c')$ passing through P''_2 . Let Y'_2 be the obtained surface.

Repeat this procedure until all E'_j with $m_j > m$ are eliminated. We shall denote the obtained surface by Y_C (Y_C depends on C). Then the successive blowings-up of Y at $\{P'_j\}$ is obtained by the successive blowings-up of Y_C as explained in Subsection 2.7, but this time every blowing-up satisfies $\mu_j := C \cdot E_{Q_j} \leq m$.

On Y_C , we have $C \equiv \nu' F - m\Omega$ where $\nu' = \nu - \sum' (m_j - m)$. Here the summation \sum' is taken over such j that $m_j > m$.

The self intersection number $C \cdot C$ on Y_C decreases from that on Y by $4m \sum' (m_j - m)$, and we have

$$(3.12) \quad \begin{aligned} \sum_j \mu_j^2 &= 4\nu' m + m^2 \omega, \\ \sum_j \mu_j &= 2\nu' + m\omega - 2, \end{aligned}$$

where $\mu_j = \min(m_j, 2m - m_j) \leq m$.

This time $4\nu' m + m^2 \omega \leq m(2\nu' + m\omega - 2)$ yields $\nu' \leq -1$. Since $C \cdot C \geq 0$ implies $\nu' \geq -\frac{m\omega}{4}$, we must have $\frac{m\omega}{4} \geq 1$, namely $m \geq \frac{4}{\omega}$.

Proposition 3.4. *The surface Y_C is a del Pezzo surface.*

Proof. We must check only $\Omega \cdot \Gamma < 0$.

Since $\Omega \cdot F = -2$ and $\Omega \cdot C \leq -2$, we can suppose that $\Gamma \notin F$ and $\Gamma \neq C$. Then $\Gamma \cdot C \geq 0$ implies $\nu' F \cdot \Gamma - m\Omega \cdot \Gamma \geq 0$, namely $\nu' F \cdot \Gamma \geq m\Omega \cdot \Gamma$. Since $\nu' \leq -1$, we have $\Omega \cdot \Gamma \leq 0$, and $\Omega \cdot \Gamma = 0$ is possible only when $F \cdot \Gamma = 0$.

Let Γ_σ be the image of Γ by the action of $\sigma \in \mathfrak{G}$. Then $\sum_\sigma \Gamma_\sigma$ is \mathfrak{G} -invariant. Since the intersection form is kept by the action of \mathfrak{G} , we have $F \cdot \sum_\sigma \Gamma_\sigma = \Omega \cdot \sum_\sigma \Gamma_\sigma = 0$, so $\sum_\sigma \Gamma_\sigma \equiv 0$ in $\text{Pic}(Y_C)$. But any principal divisor can not be an integral divisor (= positive linear combination of irreducible curves), so this is impossible, and the proposition has been proved. \square

3.10. Further blowings-up and down to reach a contradiction. For $r = 6$ or $r = 4$ (namely for $\omega = 2$ or $\omega = 4$), let F_1 be $-\frac{4}{\omega}\Omega - F$ mentioned in Theorem 2.5 (2).

We can choose F_1 and Ω as the basis of $\text{Pic}(Y_C)^\mathfrak{G}$, and we have

$$C \equiv \nu' F - m\Omega = -\nu' F_1 - \left(m + \frac{4\nu'}{\omega}\right)\Omega.$$

Put $m_1 = m + \frac{4\nu'}{\omega}$, then $-1 \geq \nu' \geq -\frac{m\omega}{4}$ yields $0 \leq m_1 < m$.

The case $m_1 = 0$ is discarded by the following reason. $m_1 = 0$ implies $C \cdot C = 0$, so $\mu_j = 0$ for any j . Let C' be the image of $s = b$, then we have $C \cdot C' = 1$ on Z' , but since $\mu_j = 0$, we have $C \cdot C' = 1$ also on Y_C . On the other hand, $F_1 \cdot F_1 = 0$ and $F_1 \cdot \Omega = -2$ imply that $F_1 \cdot D$ is even for any $D \in \text{Pic}(Y_C)^\mathfrak{G}$. This is a contradiction.

For E_j'' such that $\mu_j > m_1$, after the blowing-up E_j'' , blow down by the curve Γ_j which belongs to F_1 and passes through P_j (=base point of E_j''). (Such a curve Γ_j exists by Theorem 2.5 (1).) On the obtained surface X_1 , we have

$$C \equiv \nu_1 F_1 - m_1 \Omega \ (\nu_1 \leq -1).$$

Repeat this procedure. Since $m > m_1 > m_2 > \cdots$ is monotone decreasing, after finite steps, we reach $m_l = 0$ or $1 \leq m_l < \frac{4}{\omega}$. In the former case, $C \equiv F_l$ on X_l , so $C' \cdot C = 1$ is impossible for any other $C' \in \text{Pic}(X_l)^\oplus$. In the latter case, we can not reach $C \cdot C = 0, C \cdot \Omega = -2$ by further blowings-up.

Anyway, $k(x, y, z)$ can not be k -rational.

4. CONIC BUNDLES.

4.1. Preliminaries. First, recall that the function field of a conic bundle over \mathbb{P}_k^1 as in (1.3) may be written as $K = k(x, y, z)$ with a relation $z^2 = Py^2 + Q$ where P, Q are some non-zero separable polynomials in $k[x]$. As before, we assume that $\text{char } k \neq 2$.

Note that the rationality problem for the pair (P, Q) is equivalent with that for (Q, P) , because by putting $z = yz'$ and $y = \frac{1}{y'}$, $z^2 = Py^2 + Q$ is rewritten as $z'^2 = P + Qy'^2$. It is equivalent also with that of (P, Q') where $Q' = Q(F^2 - PG^2), F, G \in k[x]$. By putting $z = Fz' + PGy', y = Gz' + Fy', z^2 - Py^2 = Q$ is rewritten as $(F^2 - PG^2)(z'^2 - Py'^2) = Q$. When $F = 0, G = 1$, it is equivalent with that for $(P, -PQ)$.

When $\deg P = 0$ or $\deg Q = 0$ or $P/Q = \text{const.}$, the equation is reduced to the previous section. Thus we will assume $\deg P \geq 1, \deg Q \geq 1$, and $P/Q \neq c (c \in k)$.

Proposition 4.1. *If there exist $A(x), B(x), C(x) \in k[x]$ such that*

$$(4.1) \quad A^2 P + B^2 Q = C^2,$$

then $k(x, y, z)$ is $k(x)$ -rational (note that none of A, B, C is zero under the assumption above).

Proof. From (4.1), $z^2 = Py^2 + Q$ is rewritten as $B^2 z^2 = B^2 Py^2 + C^2 - A^2 P$, namely as $B^2 z^2 - C^2 = P(B^2 y^2 - A^2)$. Put $Bz + C = z'$ and $By + A = y'$, then we have $z'(z' - 2C) = Py'(y' - 2A)$. Put $z' = uy'$, then $u(u - \frac{2}{y'}C) = P(1 - \frac{2}{y'}A)$, so $y' \in k(x, u)$. This implies that $k(x, y, z) = k(x, y', z') = k(x, y', u) = k(x, u)$, so Proposition 4.1 holds. Explicitly writing, we have

$$(4.2) \quad y = \frac{-Au^2 + 2Cu - AP}{B(u^2 - P)}, \quad z = \frac{Cu^2 - 2APu + CP}{B(u^2 - P)}.$$

□

Proposition 4.2. *For a sufficiently large Galois extension l of k , $l(x, y, z)$ is $l(x)$ -rational.*

Proof. It suffices to show the existence of A, B and C in $k^{\text{sep}}[x]$ satisfying (4.1) where k^{sep} is the separable closure of k ; that is, we will show that the Hilbert norm-residue symbol $(P, Q)_2$ over the field $k^{\text{sep}}(x)$ is trivial.

Let \bar{k} be a fixed algebraic closure of k . Thus \bar{k} is a purely inseparable extension of K^{sep} .

If $\text{char } k = 0$, then $\bar{k} = k^{\text{sep}}$. By Tsen's theorem, the field $\bar{k}(x)$ is a C_1 -field [Gre69, page 22]. Hence there are polynomials $A, B, C \in \bar{k}[x]$ such that $C \neq 0$ and $(A/C)^2P + (B/C)^2Q = 1$.

Now suppose that $\text{char } k = p > 0$. Remember that $p \neq 2$. Let \mathfrak{A} be the quaternion algebra over $k^{\text{sep}}(x)$ corresponding to the Hilbert norm-residue symbol $(P, Q)_2$ over the field $k^{\text{sep}}(x)$. Since the Brauer group $\text{Br}(\bar{k}(x)) = 0$ by another theorem of Tsen [Gre69, page 4], \mathfrak{A} is split by some finite purely inseparable extension of $k^{\text{sep}}(x)$. Thus $p^n[\mathfrak{A}] = 0$ for some non-negative integer n where $[\mathfrak{A}]$ denotes the similarity class of \mathfrak{A} in the Brauer group. Because \mathfrak{A} is a quaternion algebra, it is necessary that $2[\mathfrak{A}] = 0$. Thus $[\mathfrak{A}] = 0$ and the Hilbert norm-residue symbol $(P, Q)_2$ is trivial. \square

Proposition 4.3. *If $A, B, C \in l[x]$ satisfy (4.1), then for any $f \in l[x]$, the following A_1, B_1, C_1 also satisfy (4.1):*

$$(4.3) \quad \begin{aligned} A_1 &= A(f^2 - Q), \\ B_1 &= Bf^2 + 2Cf + BQ, \\ C_1 &= Cf^2 + 2BQf + CQ. \end{aligned}$$

Proof. Regarding A_1, B_1, C_1 as quadratic polynomials of f , the comparison of the coefficients of $A_1^2P + B_1^2Q$ and C_1^2 leads to the proof. \square

Proposition 4.4. *$A, B, C \in l[x]$ in Proposition 4.3 can be chosen as the following conditions hold:*

- (1) Any two of A, B, C are mutually disjoint;
- (2) $\deg B$ is sufficiently large;
- (3) B is disjoint with Q ;
- (4) all zeros of B are simple;
- (5) $b_0 = 1$ where b_0 is the coefficient of the highest degree term of B .

Proof.

(1) Dividing by $\gcd(A, B, C)$, we can assume that A, B, C are mutually disjoint. Then any two of them are already mutually disjoint, because a common zero of two of them is necessarily a zero of the third one. Note that all zeros of P, Q are simple.

(2) Suppose that A, B, C satisfy (1). Let $f = \alpha x^n$. Then as shown below, B_1 and C_1 in (4.3) are mutually disjoint except finite number of α . Taking n so large, we get A_1, B_1, C_1 which satisfy (1) and (2). B_1 and C_1 are mutually disjoint if their resultant is not zero. The resultant is a polynomial of α (with fixed B, C, Q), so the number of zeros is finite unless the resultant is identically zero.

(3) Suppose that A, B, C satisfy (1) and (2). Let $f = \alpha \in k^\times$. Then B_1 is disjoint with Q except finite number of α . If $B(c) = Q(c) = 0$, then $B_1(c) = 2\alpha C(c) \neq 0$ for $\alpha \neq 0$. If $B(c) \neq 0, Q(c) = 0$, then $B_1(c) \neq 0$ for $\alpha \neq 0, \alpha \neq -\frac{2C(c)}{B(c)}$. This proves (3).

(4) The above B_1 has no multiple zero except finite number of α . The proof is similar with that of (2).

We can check that the resultant of B_1 and B'_1 (= the derivative of B_1) is not identically zero as a polynomial of α .

Finally dividing by a constant, we can set $b_0 = 1$. This is the claim of (5) \square

Hereafter we shall always assume the conditions (1), ..., (5) of Proposition 4.4 for A, B, C .

4.2. Biregularization of T . Let A, B, C and A_1, B_1, C_1 be as in Proposition 4.4. (Note that A_1, B_1, C_1 are arbitrary, and may not be in the form of (4.3).)

Proposition 4.5. *Let $u = \frac{Bz+C}{By+A}$ and $u_1 = \frac{B_1z+C_1}{B_1y+A_1}$. Then we have*

$$(4.4) \quad u_1 = \frac{Du + EP}{Eu + D}$$

where $D, E \in l[x]$ are mutually disjoint and $\frac{D}{E} = \frac{B_1C+BC_1}{A_1B-AB_1} = \frac{(A_1B+AB_1)P}{BC_1-B_1C}$.

Proof. Since $l(x, y, z) = l(x, u) = l(x, u_1)$, u_1 should be a linear fraction of u with $l(x)$ -coefficients.

When $u = \infty$, from (4.2) we have $y = -\frac{A}{B}, z = \frac{C}{B}$ so that $u_1 = \frac{B_1C+BC_1}{A_1B-AB_1}$. When $u = 0$, we have $y = \frac{A}{B}, z = -\frac{C}{B}$ so that $u_1 = \frac{-B_1C+BC_1}{A_1B+AB_1}$. Since $(B^2A_1^2 - B_1^2A^2)P = B^2C_1^2 - B_1^2C^2$, we have $\frac{-B_1C+BC_1}{A_1B+AB_1} = \frac{(A_1B-AB_1)P}{BC_1+B_1C}$. From these facts, we obtain $u_1 = \frac{Du+EP}{Eu+D}$ where

$$(4.5) \quad E \text{ and } D \text{ are mutually disjoint and } \frac{D}{E} = \frac{B_1C + BC_1}{A_1B - AB_1} = \frac{(A_1B + AB_1)P}{BC_1 - B_1C}.$$

□

Note that at least one of $\frac{B_1C+BC_1}{A_1B-AB_1}$ and $\frac{(A_1B+AB_1)P}{BC_1-B_1C}$ is not $\frac{0}{0}$ (i.e. at least one of $B_1C + BC_1$, $A_1B - AB_1$, $(A_1B + AB_1)P$ and $BC_1 - B_1C$ is not zero).

Let T be the birational mapping

$$(4.6) \quad T : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1, T : x \mapsto x, u \mapsto \frac{Du + EP}{Eu + D}.$$

Then $T = \text{id}$ if and only if $E = 0$, so if and only if $A = A_1, B = B_1, C = C_1$ under the assumptions (1) and (5) of Proposition 4.4. In the following discussions, we shall investigate how to blow-up $\mathbb{P}^1 \times \mathbb{P}^1$ to make T biregular.

Since T keeps x unchanged, an exceptional curve of T is in the form of $x = c$.

Proposition 4.6. *Let T be the map as in (4.6). Then $x = c (c \neq \infty)$ is an exceptional curve of T only if c is a zero of BB_1PQ .*

Proof. $x = c$ is an exceptional curve of T if and only if c is a zero of $D^2 - E^2P$. However, $D^2 - E^2P$ divides both of $(BC_1 + B_1C)^2 - (A_1B - AB_1)^2P$ and $(A_1B + AB_1)^2P^2 - (BC_1 - B_1C)^2P$, so it divides $P\{(B^2C_1^2 + B_1^2C^2) - (A_1^2B^2 + A^2B_1^2)P\} = 2B^2B_1^2PQ$. □

Remark 4.1. The curve $x = \infty$ is an exceptional curve if and only if $\deg(EP) > \deg D$.

Proposition 4.7. *Let T be the map as in (4.6).*

(1) *When $P(c) = 0$ and $Q(c) \neq 0$, $x = c$ is an exceptional curve of T if and only if $\frac{C}{B} = -\frac{C_1}{B_1}$ at $x = c$.*

(2) *When $P(c) = Q(c) = 0, Q(c) \neq 0$, $x = c$ is an exceptional curve of T if and only if $\frac{A}{B} = -\frac{A_1}{B_1}$ at $x = c$.*

Moreover, $x = c$ is mapped biregularly to the blowing up E_c of the point $(c, 0)$.

Proof. $P(c) = 0$ implies $B(c) \neq 0, B_1(c) \neq 0$ and $Q = \left(\frac{C}{B}\right)^2 = \left(\frac{C_1}{B_1}\right)^2$ at $x = c$, so $\frac{C}{B} = \pm \frac{C_1}{B_1}$ at $x = c$. If $Q(c) \neq 0$, we have $D(c) = 0$ if and only if $\frac{C}{B} = -\frac{C_1}{B_1}$ at $x = c$, hence we get (1). If $P(c) = Q(c) = 0$, none of A, A_1, B, B_1 is zero at $x = c$ but $C(c) = C_1(c) = 0$, thus we get $\left(\frac{A}{B}\right)^2 = \left(\frac{A_1}{B_1}\right)^2 = -\frac{Q}{P}$ at $x = c$, and similar discussion as above leads to the result (2).

In both cases, c is a simple zero of $D^2 - E^2P$, since $E(c) \neq 0$ and c is a simple zero of P . T maps $x = c$ to a point $(c, 0)$, so T maps $x = c$ biregularly to the blowing up E_c at $(c, 0)$ biregularly. \square

Proposition 4.8. *Let T be the map as in (4.6). When $Q(c) = 0$ and $P(c) \neq 0$, $x = c$ is an exceptional curve of T if and only if $\frac{C}{A} = -\frac{C_1}{A_1}$ at $x = c$. Moreover, $x = c$ is mapped to the blowing up E_c at the point $(c, -C(c)/A(c))$.*

Proof. $Q(c) = 0$ implies $A(c) \neq 0, A_1(c) \neq 0$ and $P = \left(\frac{C}{A}\right)^2 = \left(\frac{C_1}{A_1}\right)^2$ at $x = c$, so $\frac{C}{A} = \pm \frac{C_1}{A_1}$ at $x = c$. $D^2 - E^2P = 0$ is equivalent with $\left(\frac{C}{A}\right)^2 = \left(\frac{D}{E}\right)^2$. We see that if $\frac{C}{A} = -\frac{C_1}{A_1}$, then $\frac{D}{E} = -\frac{C}{A}$, but if $\frac{C}{A} = \frac{C_1}{A_1}$, then $\frac{D}{E} \neq \pm \frac{C}{A}$. This leads to the first statement of Proposition 4.8.

Since $D^2 - E^2P$ divides $B^2B_1^2PQ$ and BB_1 is disjoint with Q , c is a simple zero of $D^2 - E^2P$. Since T maps $x = c$ to a point $(c, -C(c)/A(c))$, this leads to the second statement of Proposition 4.8. \square

Proposition 4.9. *Let T be the map as in (4.6) and A, B, C, A_1, B_1, C_1 be as in Proposition 4.4. When c is a zero of BB_1 , $x = c$ is an exceptional curve of T if and only if*

- (1) *only one of $B(c)$ or $B_1(c)$ is zero, or*
- (2) *$B(c) = B_1(c) = 0$ and $\frac{C}{A} = -\frac{C_1}{A_1}$ at $x = c$.*

In case (1), $x = c$ is mapped to the blowing up at $(c, -C(c)/A(c))$ or $(c, C_1(c)/A_1(c))$. In case (2), $x = c$ is mapped to the blowing up of order 2 at $(c, -C(c)/A(c))$.

Proof. If $B(c) = 0$ and $B_1(c) \neq 0$, then neither of A nor C is zero and $P = \left(\frac{C}{A}\right)^2$ at $x = c$. On the other hand, $\frac{D}{E} = \frac{B_1C + BC_1}{A_1B - AB_1} = -\frac{C}{A}$ at $x = c$. From this, c is a simple zero of $D^2 - E^2P$ (simplicity comes from that c is a simple zero of B) and we get (1).

If $B(c) \neq 0$ and $B_1(c) = 0$, similar discussions hold, replacing $-C/A$ by C_1/A_1 .

If $B(c) = B_1(c) = 0$, we have $P = \left(\frac{C}{A}\right)^2 = \left(\frac{C_1}{A_1}\right)^2$ at $x = c$, so $\frac{C}{A} = \pm \frac{C_1}{A_1}$ at $x = c$. When $\frac{C}{A} = -\frac{C_1}{A_1}$ we have $\frac{D}{E} = -\frac{C}{A}$ at $x = c$, and c is a double zero of $D^2 - E^2P$. Thus $x = c$ is an exceptional curve of T and T maps $x = c$ to the blowing up of order 2 at $(c, -C(c)/A(c))$. The fundamental point of T is $(c, C(c)/A(c))$. Then T maps E'_c , the blowing-up at this point, to E_c , the blowing up at the image point. When $\frac{C}{A} = \frac{C_1}{A_1}$ at $x = c$, we have $\frac{D}{E} \neq \pm \frac{C}{A}$ at $x = c$, and $x = c$ is not an exceptional curve of T . \square

The curve $x = \infty$ may or may not be an exceptional curve of T , but we have the following:

Proposition 4.10. *Let T be the map as in (4.6) and F_∞ be the blowing up of order $\lfloor d_P/2 \rfloor = \max\{m \in \mathbb{Z} \mid m \leq d_P/2\}$ at $(\infty, \infty) \in \mathbb{P}^1 \times \mathbb{P}^1$. Then T maps F_∞ to F_∞*

except the following three cases:

- (1) d_P is even, d_Q is odd and $\frac{a_0}{c_0} = -\frac{a_{1,0}}{c_{1,0}}$;
- (2) d_P is odd, d_Q is even and $c_0 = -c_{1,0}$;
- (3) d_P is odd, d_Q is odd and $a_0 = -a_{1,0}$.

Here, a_0 (resp. $c_0, a_{1,0}, c_{1,0}$) is the coefficient of the highest degree term of A (resp. C, A_1, C_1). In these three cases, T maps F_∞ to E_∞ , once more blowing up of F_∞ .

Proof. Let $r = \deg D - \deg E$. By checking the order of infinity at $x = \infty$ of u and $u_1 = \frac{Du+EP}{Eu+D}$, we see that when d_P is odd, F_∞ is mapped to F_∞ if $r > \frac{d_P}{2}$ and to E_∞ if $r < \frac{d_P}{2}$.

When d_Q is even, $d_C - d_A > d_P/2$ and $c_0^2 = c_{10}^2 = q_0$. From this we see that $r < \frac{d_P}{2}$ if and only if $c_0 = -c_{10}$. When d_Q is odd, $d_C - d_A < d_P/2$ and $a_0^2 = a_{10}^2 = -q_0/p_0$. From this we see that $r < \frac{d_P}{2}$ if and only if $a_0 = -a_{10}$. Note that $a_0^2 = a_{10}^2$ always.

When d_P is even, T maps F_∞ to F_∞ whenever $r \neq \frac{d_P}{2}$. If $r = \frac{d_P}{2}$, $u \sim \lambda x^r$ implies $u_1 \sim \frac{\lambda d_0 + c_0 p_0}{\lambda e_0 + d_0} x^r$, so that T maps F_∞ to F_∞ unless $d_0^2 - e_0^2 p_0 = 0$. But we can verify that $r = \frac{d_P}{2}$ and $d_0^2 - e_0^2 p_0 = 0$ are equivalent with that d_Q is odd and $\frac{a_0}{c_0} = -\frac{a_{10}}{c_{10}}$. In this case, we need once more blowing up, and T maps F_∞ to E_∞ . \square

4.3. Construction of Y . For a given A, B, C , $u = \frac{By+A}{Bz+C}$ is mapped to $\frac{B^\sigma y + A^\sigma}{B^\sigma z + C^\sigma}$ by the action of $\sigma \in \mathfrak{G} = \text{Gal}(l/k)$. Here A^σ is the polynomial obtained from A by replacing all coefficients to its conjugates by σ .

We shall construct a non-singular projective surface Y on which \mathfrak{G} acts in a Zariski homeomorphic way. Starting from $\mathbb{P}^1 \times \mathbb{P}^1$, we repeat blowings-up and blowings-down.

The automorphism Φ_σ of $l(x, u)$ is induced by the point transformation Ψ_σ of $\mathbb{P}^1 \times \mathbb{P}^1$ as $(\Phi_\sigma f)(x, y) = \left(f(\Psi_\sigma^{-1}(x, y)) \right)^\sigma$. Here $\Psi_\sigma = \tau_\sigma \circ \tilde{\sigma}$, $\tilde{\sigma} : (x, u) \mapsto (x^\sigma, u^\sigma)$ and $\tau_\sigma : (x, u) \mapsto (x, \frac{D_\sigma u - E_\sigma P}{-E_\sigma u + D_\sigma})$, where

$$E_\sigma \text{ and } D_\sigma \text{ are mutually disjoint and } \frac{D_\sigma}{E_\sigma} = \frac{BC^\sigma + B^\sigma C}{A^\sigma B - AB^\sigma} = \frac{(A^\sigma B + AB^\sigma)P}{BC^\sigma - B^\sigma C}.$$

Proposition 4.11. *The Galois group \mathfrak{G} acts on some Y_{rs} in a Zariski homeomorphic way. (Y_{rs} is defined in Subsection 2.9, and r and s are given in the proof.)*

Proof. $\Psi_\sigma = \tau_\sigma \circ \tilde{\sigma}$ is Zariski homeomorphic except on the line $x = c$, where c is a zero of P or a zero of Q or a conjugate of a zero of B or $c = \infty$.

Let $s = s_1 + s_2 + s_3 + s_4$ where s_1 (resp. s_2 , resp. s_3) is the number of $c \in k^{\text{sep}}$ such that $P(c) = 0$ and $Q(c) \notin k(c)^2$ (resp. $Q(c) = 0$ and $P(c) \notin k(c)^2$, resp. $P(c) = Q(c) = 0$ and $-\frac{Q}{P}(c) \notin k(c)^2$), and let E_c be the blowing up at $(c, 0)$ (resp. $(c, C(c)/A(c))$, resp. $(c, 0)$). One more E at (∞, ∞) is added for three cases (i), (ii), (iii) stated later. So, $s_4 = 1$ for these cases and 0 otherwise.

Suppose that $P(c) = 0$ and $Q(c) \notin k(c)^2$. Let $H_c = \{\sigma \in \mathfrak{G} | c^\sigma = c\}$. Since $Q(c) = \left(\frac{C(c)}{B(c)}\right)^2 \in k(c) \setminus k(c)^2$, we see that $\frac{C}{B}(c) = \frac{C^\sigma}{B^\sigma}(c)$ for half ones of $\sigma \in H_c$ and $\frac{C}{B}(c) = -\frac{C^\sigma}{B^\sigma}(c)$ for other half ones.

From this, we see that for any conjugates c' and c'' of c , half ones of Ψ_σ such that $c'^\sigma = c''$ map $x = c'$ to $x = c''$, $E_{c'}$ to $E_{c''}$ and other half ones of Ψ_σ map

$x = c'$ to $E_{c''}$, $E_{c'}$ to $x = c''$. Note that on the blown up surface, the class of $x = c'$ is $F - E_{c'}$. So the number of Ψ_σ which map $E_{c'}$ to $E_{c''}$ is the same with the number of Ψ_σ which map $E_{c'}$ to $F - E_{c''}$.

Let $r = r_1 + r_2 + r_3 + r_4 + r_5$. Here $r_5 = \deg P/2$ or $(\deg P - 1)/2$ according to whether $\deg P$ is even or odd. $r_4 = \deg B$ and r_1, r_2, r_3 are given below.

Suppose that $P(c) = 0, Q(c) \neq 0, Q(c) \in k(c)^2$. Then $Q(c) = \left(\frac{C(c)}{B(c)}\right)^2 \in k(c)^2$ implies $\frac{C}{B}(c) = \frac{C^\sigma}{B^\sigma}(c)$ for all $\sigma \in H_c$. From this, we see that for any conjugates of c' and c'' of c , all Ψ_σ such that $c'^\sigma = c''$ map $x = c'$ to $x = c''$, $E_{c'}$ to $E_{c''}$ or all Ψ_σ map $x = c'$ to $E_{c''}$, $E_{c'}$ to $x = c''$. So, we can divide such c into two blocks, so that if c' and c'' are in the same block, all Ψ_σ map $x = c'$ to $x = c''$, $E_{c'}$ to $E_{c''}$ and if c'' belongs to the different block with c' , all Ψ_σ map $x = c'$ to $E_{c''}$, $E_{c'}$ to $x = c''$.

Let r_1 be the number of c in the second block. When c is in the first block, let F_c be $(x = c)$. When c is in the second block, let F_c be the blowing down of E_c by $(x = c)$. Then Ψ_σ maps always $F_{c'}$ to $F_{c''}$. Note that on the blown-up and down surface, the class of F_c is F . So Ψ_σ induces the transformation in the same class F .

The same discussions hold for $Q(c) = 0, P(c) \neq 0, P(c) \in k(c)^2$ and also for $P(c) = Q(c) = 0, -\frac{Q}{P}(c) \in k(c)^2$.

When $B(c) = 0$, (then $P(c) \neq 0$ and $Q(c) \neq 0$), let F_c be the blowing up at $(c, C(c)/A(c))$, blown down by $(x = c)$ afterwards. When c is a conjugate of a zero of B and $B(c) \neq 0$, let F_c be $(x = c)$. Then Ψ_σ map $F_{c'}$ to $F_{c''}$ always. Thus Ψ_σ induces an automorphism of Pic which keeps the class F unchanged.

For $x = \infty$, let F_∞ be the blowing up of order $\deg P/2$ or $(\deg P - 1)/2$ at (∞, ∞) , blown down by E_j (=blowings up of smaller orders) afterwards. (See the discussions in Subsection 3.3.) Then the class of F_∞ is F . Ψ_σ maps F_∞ to F_∞ except the following three cases:

- (i) $\deg P$ even, $\deg Q$ odd and $p_0 \notin k^2$;
- (ii) $\deg P$ odd, $\deg Q$ even and $q_0 \notin k^2$;
- (iii) $\deg P$ odd, $\deg Q$ odd and $-q_0/p_0 \notin k^2$.

In these cases, take once more blowing-up E_∞ of F_∞ , then half ones of $\sigma \in \mathfrak{G}$ map E_∞ to E_∞ and other half ones map E_∞ to F_∞ .

The results are summarized as follows. The blowings up are only E_c such that $P(c) = 0, Q(c) \notin k(c)^2$, or $Q(c) = 0, P(c) \notin k(c)^2$ or $P(c) = Q(c) = 0, -\frac{P}{Q}(c) \notin k(c)^2$. The blowings up and down afterwards are at other zeros of BPQ . (For the zeros of PQ , only at c in the second block.)

For $x = \infty$, the blowings up and down F_∞ is added, and the blowing up E_∞ is added in some cases ((i), (ii), (iii) mentioned above).

Thus we reach the desired surface Y_{rs} . □

4.4. Pic(Y) and Pic(Y) $^\mathfrak{G}$. From the discussion in the previous subsection, we can determine Pic(Y) and Pic(Y) $^\mathfrak{G}$ as follows.

Pic(Y) is of rank $s + 2$ as a \mathbb{Z} -module with the basis E_i ($1 \leq i \leq s$), F and F' . The action of $\sigma \in \mathfrak{G}$ is represented as the following matrix with the basis above in this order:

$$(4.8) \quad g_\sigma = \begin{pmatrix} A_\sigma & X_\sigma & 0 \\ 0 & 1 & 0 \\ U_\sigma & \alpha_\sigma & 1 \end{pmatrix}.$$

Here A_σ is an $s \times s$ matrix whose entries are 0, 1 or -1 . The matrix obtained by replacing all the entries -1 's of A_σ by 1 is the permutation matrix which represents the permutation of c_i induced by σ . For any (i, j) , $1 \leq i, j \leq s$, the number of A_σ where (i, j) -entry is 1 is equal with the number of A_σ whose (i, j) -entry is -1 .

X_σ is a column vector whose entries are 0 or 1. The j -th entry is 0 or 1 according to whether the only one non-zero entry of A_σ in j -th row is 1 or -1 .

U_σ is a row vector whose entries are -1 or 0. The i -th entry is -1 or 0 according to whether $(x = c_i)$ is an exceptional curve of T_σ or not, so according to whether the only one non-zero entry of A_σ in i -th column is -1 or 1.

α_σ is some integer. Its value is determined in Subsection 4.7. From (4.8), $\sum_\sigma g_\sigma$ is calculated as follows:

$$(4.9) \quad \sum_\sigma g_\sigma = \frac{|\mathfrak{G}|}{2} \begin{pmatrix} O_s & 1 & 0 \\ 0 & 2 & 0 \\ -1 & \alpha & 2 \end{pmatrix}$$

where O_s is the $s \times s$ matrix whose all entries are zero. $\frac{|\mathfrak{G}|}{2}\alpha = \sum_\sigma \alpha_\sigma$, so α may not be an integer.

This matrix is the same with (3.6) except the $(s+2, s+1)$ entry. (But r in (3.6) is replaced by s here.) So, Proposition 3.3 holds. For the sake of convenience, we shall write it again.

Proposition 4.12. *The submodule $\text{Pic}(Y)^\mathfrak{G}$ of \mathfrak{G} -invariant classes has rank 2 with the basis F and Ω . We have $F \cdot F = 0$, $F \cdot \Omega = -2$ and $\Omega \cdot \Omega = 8 - s$.*

Remark 4.2. Proposition 4.12 holds for the case $s > 0$. For the case $s = 0$, $\text{Pic}(Y)$ is of rank 2 and every class is \mathfrak{G} -invariant.

4.5. Irrationality for $s \geq 6$ and $s = 4$. The discussions in Subsection 3.7 to Subsection 3.10 rely only on the structure of $\text{Pic}(Y)^\mathfrak{G}$. Since $\text{Pic}(Y)^\mathfrak{G}$ is the same, the discussions there can be applied to prove the irrationality of $k(x, y, z)$ (but r in Subsection 3.7 to Subsection 3.10 should be replaced by s here).

By the discussions in Subsection 3.7 and Subsection 3.8, $k(x, y, z)$ is not k -rational when $s \geq 8$. By the discussions in Subsection 3.9 and Subsection 3.10, $k(x, y, z)$ is not k -rational when $s = 6$ or 4 . This argument can be applied also for $s = 7$, putting $\omega = 1$ instead of $\omega = 2$ or 4 . See Theorem 2.5.

Only remained are the proof of irrationality for $s = 5$ and the proof of rationality for $s \leq 3$.

4.6. The case $s = 5$. Assume that $k(x, y, z)$ is k -rational, and we shall derive a contradiction. Let Y' be the del Pezzo surface obtained from Y (see Subsection 3.9). On the surface Y' , the class $D = -F - \Omega_{Y'}$ satisfies $D \cdot D = D \cdot \Omega_{Y'} = -1$. Similarly as Theorem 2.5 and Theorem 2.6, we can verify that there exists a unique irreducible curve L which belongs to D . Let Y'' be the blowing down of Y' by L . Then, both of F and $-\Omega_{Y'}$ in $\text{Pic}(Y')$ are mapped to $-\Omega_{Y''}$ in $\text{Pic}(Y'')$. Thus, $\text{Pic}(Y'')^\mathfrak{G}$ has rank 1 with $\Omega_{Y''}$ as its basis.

We shall write Y'' as Y . The problem is rewritten on the surface Y as follows. Y is a del Pezzo surface with $\omega = 4$. \mathfrak{G} acts Zariski homeomorphic on Y and $\text{Pic}(Y)^\mathfrak{G}$ has rank 1 with the basis Ω .

If $k(x, y, z)$ is k -rational, there exist \mathfrak{G} -invariant irreducible curves C, C' on Y such that we can reach $C \cdot C = 0, C \cdot \Omega = -2, C \cdot C' = 1$ after some blowings-up $\{E'_j\}$.

Suppose $C \equiv -m\Omega$ in $\text{Pic}(Y)$, then $C \cdot C = 4m^2$ and $C \cdot \Omega = -4m$. So we have

$$\sum_j m_j^2 = 4m^2, \sum_j m_j = 4m - 2$$

where $m_j = C \cdot E'_j$. Especially we have $m > 0$.

Proposition 4.13. *Put $m_1 = \max_j m_j$. Then*

- (1) $m_1 > m$,
- (2) *the number of j such that $m_j = m_1$ is at most 3.*

Proof.

- (1) It suffices to show that $\sum m_j^2 / \sum m_j > m$, namely $4m^2 / (4m - 2) > m$. However, $\frac{4m^2}{4m-2} = m \frac{4m}{4m-2} > m$ is evident.
- (2) Suppose that the number of the desired j is q , then we have $qm_1 \leq \sum m_j = 4m - 2$, so that $q \leq \frac{4m-2}{m_1} < \frac{4m_1-2}{m_1} < 4$. \square

Since the family of the blowings-up Φ commutes with the action of \mathfrak{G} , the configuration of $\{E'_j\}$ is \mathfrak{G} -invariant, and the action of \mathfrak{G} induces a permutation of j with $m_j = m_{\sigma(j)}$ since C is \mathfrak{G} -invariant. This implies that the base point P_1 of E'_1 has at most three conjugates including P_1 itself.

Proposition 4.14. *When $s = 5$, $k(x, y, z)$ is not k -rational.*

Proof. We split the proof into three cases:

(I) When P_1 is a \mathfrak{G} -invariant point.

Let \tilde{Y} be the blowing-up of Y at P . We have $\Omega_{\tilde{Y}} \cdot \Omega_{\tilde{Y}} = 3$. We shall show that \tilde{Y} is a del Pezzo surface. Let Γ be an irreducible curve on \tilde{Y} . If $\Gamma = E'_1$, then $\Gamma \cdot \Omega_{\tilde{Y}} = -1$. Other Γ comes from an irreducible curve on Y , and $\Gamma \cdot \Omega_{\tilde{Y}} = \Gamma \cdot \Omega_Y + m'_1$ where $m'_1 = \Gamma \cdot E'_1$. Suppose that $\Gamma \cdot \Omega_Y = -\alpha, \alpha > 0$, then $\Gamma \cdot C = m\alpha$ and $C \cdot E'_1 = m_1$ imply that $m\alpha \geq m_1 m'_1$, so that $m'_1 \leq \frac{m\alpha}{m_1} < \alpha$, which means $\Gamma \cdot \Omega_{\tilde{Y}} < 0$. Thus \tilde{Y} is a del Pezzo surface.

$\text{Pic}(\tilde{Y})^{\mathfrak{G}}$ has rank 2 with $\Omega_{\tilde{Y}}$ and E'_1 as its basis. Put $F = -E'_1 - \Omega_{\tilde{Y}}$, then $F \cdot F = 0, F \cdot \Omega_{\tilde{Y}} = -2$ and $\text{Pic}(\tilde{Y})^{\mathfrak{G}}$ is generated by F and $\Omega_{\tilde{Y}}$. Since $\Omega_{\tilde{Y}} = \Omega_Y + E'_1$, we have $C \equiv -m\Omega_{\tilde{Y}} - (m_1 - m)E'_1 = (m_1 - m)F - (2m - m_1)\Omega_{\tilde{Y}}$. If $2m - m_1 = 0$, then $C \cdot C = 0$ on \tilde{Y} and $C \cdot C' = 1$ can not happen for other \mathfrak{G} -invariant irreducible curve C' as mentioned in Subsection 3.10. Otherwise $2m - m_1 > 0$. For all $j \geq 2$ such that $m_j > 2m - m_1$, after blowing-up E'_j , blow down by the irreducible curve which belongs to F and passes through the base point of E'_j . On the obtained surface $\bar{\tilde{Y}}$, we have $C \equiv \nu F - (2m - m_1)\Omega$ with $\nu \leq -1$.

Let Z be the blow down of $\bar{\tilde{Y}}$ by the irreducible curve belonging to $-F - \Omega$. Then Z is a del Pezzo surface with $\omega = 4$, and we have $C \equiv -(2m - m_1 + \nu)\Omega$ in $\text{Pic}(Z)$. This is the same situation with the original Y , but m is replaced by $\mu = 2m - m_1 + \nu < m + \nu < m$.

(II) When P_1 and P_2 are mutually conjugate.

Let $\tilde{\tilde{Y}}$ be the blowing-up of Y at P_1 and P_2 . We have $\Omega_{\tilde{\tilde{Y}}} \cdot \Omega_{\tilde{\tilde{Y}}} = 2$, and $\tilde{\tilde{Y}}$ is a del Pezzo surface by the same reason as (I). \mathfrak{G} acts Zariski homeomorphic on $\tilde{\tilde{Y}}$ and $\text{Pic}(\tilde{\tilde{Y}})^{\mathfrak{G}}$ has rank 2 with $\Omega_{\tilde{\tilde{Y}}}$ and $E'_1 + E'_2$ as its basis. Since $\Omega_{\tilde{\tilde{Y}}} = \Omega_Y + E'_1 + E'_2$,

we have

$$C \equiv -m\Omega_{\tilde{Y}} - (m_1 - m)(E'_1 + E'_2).$$

By Theorem 2.6, for $i = 1, 2$, there exists a unique irreducible curve L_i such that $L_i \equiv -E'_i - \Omega_{\tilde{Y}}$ and $L_i \cdot L_i = L_i \cdot \Omega_{\tilde{Y}} = -1$. Let Z be the blowing down of \tilde{Y} by L_1 and L_2 . Since all of E'_1, E'_2 and $-\Omega_{\tilde{Y}}$ in $\text{Pic}(\tilde{Y})$ are mapped to $-\Omega_Z$ in $\text{Pic}(Z)$, we have $C \equiv -(3m - 2m_1)\Omega_Z$ in $\text{Pic}(Z)$. This is the same situation with the original Y , but m is replaced by $\mu = 3m - 2m_1 < m$.

(III) When P_1, P_2 and P_3 are mutually conjugate.

Let $\tilde{\tilde{Y}}$ be the blowing-up of Y at P_1, P_2 and P_3 . We have $\Omega \cdot \Omega = 1$ on $\tilde{\tilde{Y}}$, and $\tilde{\tilde{Y}}$ is a del Pezzo surface by the same reason as (I). \mathfrak{G} acts Zariski homeomorphic on $\tilde{\tilde{Y}}$ and $\text{Pic}(\tilde{\tilde{Y}})^{\mathfrak{G}}$ has rank 2 with Ω and $E'_1 + E'_2 + E'_3$ as its basis.

By Theorem 2.6, for $i = 1, 2, 3$, there exists a unique irreducible curve L_i such that $L_i \equiv -E'_i - 2\Omega$ and $L_i \cdot L_i = L_i \cdot \Omega = -1$. Let Z be the blowing-down of $\tilde{\tilde{Y}}$ by L_1, L_2 and L_3 . Then we have $C \equiv -(7m - 6m_1)\Omega_Z$ in $\text{Pic}(Z)$, and $\mu = 7m - 6m_1 < m$.

In any cases of (I), (II) and (III), we can replace Y with Z with a smaller value of m . Repeat this procedure. After finite steps, we reach $m \leq 0$, which means that $C \cdot C = 0, C \cdot \Omega = -2$ can not be reached by any blowings-up $\{E'_j\}$. \square

4.7. Impossibility of $s = 1$. Consider the action of \mathfrak{G} on $\text{Pic}(Y)$.

The image of $\Psi_\sigma - \text{id}$ should be contained in the kernel of $\sum_\sigma \Psi_\sigma$. $\Psi_\sigma - \text{id}$ maps F' to $-\sum' E_i + \alpha_\sigma F$ for some integer α_σ , the sum \sum' being taken over such i that $(x = c_i)$ is an exceptional curve of τ_σ .

$\sum \Psi_\sigma$ maps E_i to $\frac{|\mathfrak{G}|}{2}F$ ($1 \leq i \leq s$), and F to $|\mathfrak{G}|F$, so $-\sum' E_i + \alpha_\sigma F$ to $\frac{|\mathfrak{G}|}{2}(-n_\sigma + 2\alpha_\sigma)F$ where n_σ is the number of i such that $(x = c_i)$ is an exceptional curve of τ_σ .

From this we see that $n_\sigma = 2\alpha_\sigma$, so n_σ is even.

Suppose that $s \neq 0$, and $x = c_1$ is an exceptional curve of τ_σ . Then, since $n_\sigma \geq 2$, there exists at least one more exceptional curve, so that $s \geq 2$. This shows that $s \neq 0$ implies $s \geq 2$, so the case $s = 1$ never happens.

4.8. The case where $s = 0$. The action $\tau_\sigma \circ \tilde{\sigma}$ is Zariski homeomorphic on Y , which is obtained from $\mathbb{P}^1 \times \mathbb{P}^1$ by r -times blowing up and down. $\text{Pic}(Y)$ is of rank 2 with the basis F, F' . We have $F \cdot F = 0, F \cdot F' = 1, F' \cdot F' = r$. Every class is \mathfrak{G} -invariant.

Proposition 4.15. *If there exists a \mathfrak{G} -invariant curve linear in u , namely $G_1(x)u + G_2(x) = 0$, then $l(x, u) = l(x, w)$ with some \mathfrak{G} -invariant w .*

Proof. Put $v = G_1(x)u + G_2(x)$, then (x, v) is a transcendent basis of $l(x, u)$, and the curve $v = 0$ is \mathfrak{G} -invariant. Since τ_σ keeps x invariant, τ_σ maps v to a linear fraction of v with $l(x)$ -coefficients, namely

$$\tau_\sigma : v \mapsto \frac{\alpha_\sigma(x)v + \beta_\sigma(x)}{\gamma_\sigma(x)v + \delta_\sigma(x)}, \alpha_\sigma, \beta_\sigma, \gamma_\sigma, \delta_\sigma \in l[x].$$

But since $v = 0$ is invariant, we have $\beta_\sigma = 0$ for any σ . Put $v' = \frac{1}{v}$, then

$$\tau_\sigma : v' \mapsto \frac{\delta_\sigma(x)v' + \gamma_\sigma(x)}{\alpha_\sigma(x)}.$$

Namely $\tau_\alpha : v' \mapsto \alpha_\sigma(x)v' + \beta_\sigma(x), \alpha_\sigma, \beta_\sigma \in l(x)$. From this we see that some $w = \alpha(x)v' + \beta(x)$ ($\alpha, \beta \in l(x), \alpha \neq 0$) is \mathfrak{G} -invariant (see [HK95]). Thus we obtained a \mathfrak{G} -invariant transcendent basis (x, w) . \square

Proposition 4.16. *When $s = 0$, $k(x, y, z)$ is k -rational except the following case. Both of $\deg P$ and $\deg Q$ are even and $a^2p_0 + b^2q_0 = c^2$ has no non-zero solution (a, b, c) in k .*

Proof. Assume that some class $\nu F + F'$ with $\nu < -\frac{r}{2}$ contains an irreducible curve Γ , then $\Gamma \cdot \Gamma = 2\nu + r < 0$, so that Γ is \mathfrak{G} -invariant because Γ is the unique irreducible curve in this class. Therefore $k(x, y, z)$ is k -rational by Proposition 4.15.

When r is odd, such ν exists. Let $d = \frac{r-1}{2}$. Then $G_1(x)u + G_2(x)$ with $\deg G_1, \deg G_2 \leq d$ has $2(d+1) = r+1$ coefficients. The condition that $G_1(x)u + G_2(x) = 0$ passes through r points yields r linear equations on these coefficients, so there exists a non-zero solution. Suppose that the curve passes through r blown up points, then $\nu = d - r = \frac{r-1}{2} - r = -\frac{r+1}{2} < -\frac{r}{2}$. Since $\deg B$ is sufficiently large by Proposition 4.4 (2), we have $\deg B > d$, so neither G_1 nor G_2 is zero. If G_1 and G_2 are not mutually disjoint, divide by GCD to get an irreducible curve. Thus when r is odd, $k(x, y, z)$ is k -rational.

Suppose that r is even and let $d = \frac{r}{2}$. Similar discussion as above shows that the curves $G_1(x)u + G_2(x) = 0$ with $\deg G_1, \deg G_2 \leq d$ which pass through all r blown up points form a vector space of at least two dimensional.

If the dimension is 3 or more, there exists a solution among them such that the coefficients of the highest degree terms of G_1 and G_2 are zero, and for such a solution, we have $\nu < d - r = \frac{r}{2} - r = -\frac{r}{2}$, so the problem is reduced to the solved case.

Suppose that the desired solutions are two dimensional, then the family of desired curves is parametrized by \mathbb{P}^1 . They are mutually disjoint in Y because of $\Gamma \cdot \Gamma = 0$, and their union covers all Y . The relation that Γ passes through P defines a one-to-one correspondence between a point P on F_∞ and a curve Γ in this family, because of $\Gamma \cdot F_\infty = 1$.

Consider the action of Ψ_σ on F_∞ . If there exists a \mathfrak{G} -invariant point on F_∞ , then the corresponding curve is \mathfrak{G} -invariant, so $l(x, u)$ has a transcendent basis (x, w) with some \mathfrak{G} -invariant w .

On the contrary, if \mathfrak{G} -invariant point does not exist on F_∞ , then \mathfrak{G} -invariant Γ does not exist, so \mathfrak{G} -invariant point does not exist on Y at all, because the curves are mutually disjoint. Hence $l(x, u)$ can never have a \mathfrak{G} -invariant transcendent basis. Thus the rationality of $k(x, y, z)$ over k is equivalent with the existence of \mathfrak{G} -invariant point on F_∞ .

If $\deg P$ is odd and $\deg Q$ is even, then $s = 0$ implies $q_0 \in k^2$, so that $c_0 = \sqrt{q_0} \in k$ and τ_σ on F_∞ is given by $\lambda \mapsto \lambda + \frac{a_0 - a_0^\sigma}{2c_0} p_0$. So $\lambda = \frac{a_0 p_0}{2c_0}$ is invariant by $\Phi_\sigma = \tau_\sigma \circ \tilde{\sigma}$ for all $\sigma \in \mathfrak{G}$. If both of $\deg P$ and $\deg Q$ are odd, similar arguments show that $\lambda = \frac{c_0}{2a_0}$ is \mathfrak{G} -invariant.

If $\deg P$ is even and $\deg Q$ is odd, we have $p_0 \in k^2$ and $\frac{c_0}{a_0} = \sqrt{p_0} \in k$. τ_σ on F_∞ is given by $\lambda \mapsto \frac{(c+c^\sigma)\lambda + (a-a^\sigma)p_0}{(a-a^\sigma)\lambda + (c+c^\sigma)}$, so $\lambda = \sqrt{p_0}$ is \mathfrak{G} -invariant.

Assume that both of $\deg P$ and $\deg Q$ are even. The rationality of $k(x, y, z)$ is independent of the choice of A, B, C such that $A^2P + B^2Q = C^2$, so we can choose convenient ones.

Let $a_0 = 0, c_0 = \sqrt{q_0}$, then $c_0^\sigma = \pm c_0$. If $q_0 \in k^2$, τ_σ on F_∞ is the identity for all $\sigma \in \mathfrak{G}$, so every $\lambda \in k$ is \mathfrak{G} -invariant. If $\sqrt{q_0} \notin k$, every \mathfrak{G} -invariant λ must belong to $k(c_0) = k(\sqrt{q_0})$.

If $c_0^\sigma = -c_0$, we have $d_{\sigma 0} = 0$, so τ_σ is $\lambda \mapsto \frac{p_0}{\lambda}$ on F_∞ . Let $\lambda = \lambda_1 + c_0\lambda_2$, $\lambda_1, \lambda_2 \in k$, and $\bar{\lambda} = \lambda_1 - c_0\lambda_2$. Then λ is \mathfrak{G} -invariant if and only if $\lambda\bar{\lambda} = p_0$, namely $\lambda_1^2 - q_0\lambda_2^2 = p_0$. The existence of such λ_1, λ_2 is equivalent with the existence of non-zero $(a, b, c) \in k \times k \times k$ such that $a^2p_0 + b^2q_0 = c^2$. \square

4.9. The case where $s = 2$. The surface Y , on which \mathfrak{G} acts in a Zariski homeomorphic way, is of type Y_{r2} . $\text{Pic}(Y)$ is of rank 4 with the basis F, F', E_1 and E_2 . The intersection form is $F \cdot F = 0, F \cdot F' = 1, F' \cdot F' = r, F \cdot E_i = F' \cdot E_i = 0, E_i \cdot E_i = -1$ ($i = 1, 2$) and $E_1 \cdot E_2 = 0$.

First, we shall show that there exists an irreducible curve Γ such that $\Gamma \cdot \Gamma < 0$ and $\Gamma \cdot F = 1$.

Let Y_{r0} be the surface before the blowings-up E_1 and E_2 . The discussions in the proof of Proposition 4.16 show the followings.

When r is odd, there exists an irreducible curve Γ such that $\Gamma \cdot \Gamma < 0$ and $\Gamma \cdot F = 1$ on Y_{r0} . Since $\Gamma \cdot \Gamma$ decreases by any blowing-up, we have $\Gamma \cdot \Gamma < 0$ on Y .

When r is even, there exists a family of irreducible curves such that $\Gamma \cdot \Gamma = 0$ and $\Gamma \cdot F = 1$ on Y_{r0} . Among them, there exists a curve Γ which passes through P_1 (= the base point of E_1), and we have $\Gamma \cdot \Gamma < 0$ on Y .

Proposition 4.17. *There exist exactly two irreducible curves Γ_1 and Γ_2 which satisfy $\Gamma_i \cdot \Gamma_i < 0$ and $\Gamma_i \cdot F = 1$ ($i = 1, 2$). They are mutually conjugate, and their classes are*

$$(4.10) \quad \Gamma_1 : F' - \frac{r}{2}F - E_1, \Gamma_2 : F' - \frac{r}{2}F - E_2$$

when r is even.

The surface Y_{r0} (before the blowings-up E_1, E_2) is biregular with $\mathbb{P}^1 \times \mathbb{P}^1$.

The odd r case can never happen.

Proof. We choose $\sigma \in \mathfrak{G}$ so that it maps E_1 to $F - E_1$. Then it maps E_2 to $F - E_2$ (note that $n_\sigma = 2$ as stated in Subsection 4.7), and maps F' to $F' + F - E_1 - E_2$ because F and $\Omega = -2F' + (\frac{r}{2} - 2)F + E_1 + E_2$ are invariant. So σ maps $F' + \nu F$ to $F' + (\nu + 1)F - E_1 - E_2$ and maps $F' + \nu F - E_1$ to $F' + \nu F - E_2$.

Suppose that an irreducible curve Γ_1 belongs to $F' + \nu F - E_1$, then its conjugate Γ_2 belongs to $F' + \nu F - E_2$. Since $0 \leq \Gamma_1 \cdot \Gamma_2 = \Gamma_1 \cdot \Gamma_1 + 1$, we have $\Gamma_1 \cdot \Gamma_1 = -1$ and $\nu = -\frac{r}{2}$, r being even.

Let v be the ratio of the defining equation of Γ_1 and Γ_2 :

$$(4.11) \quad v = \frac{f_2(x)u + g_2(x)}{f_1(x)u + g_1(x)},$$

$$\Gamma_i : f_i(x)u + g_i(x) = 0.$$

Then, since $\Gamma_1 \cdot \Gamma_2 = 0$, v is not $\frac{0}{0}$ at any point of Y , so the birational mapping $(x, u) \mapsto (x, v)$ is regular from Y to $\mathbb{P}^1 \times \mathbb{P}^1$. Since it is injective on Y_{r0} , it defines a biregular mapping from Y_{r0} to $\mathbb{P}^1 \times \mathbb{P}^1$. On the surface Y , there exist two exceptional curves E_1 and E_2 , and E_i is mapped to ε_i , where ε_1 is the blowing up at (c_1, ∞) and ε_2 is the blowing up at $(c_2, 0)$.

σ maps $(v = 0)$ to $(v = \infty)$, so τ_σ is written in terms v as

$$(4.12) \quad \tau_\sigma : v \mapsto \frac{\varphi(x)}{v}, \quad \varphi(x) \in l(x).$$

But τ_σ is regular on $x \neq c_1, c_2$, so $\varphi(x)$ has a pole at c_1 and a zero at c_2 , so that $\varphi = \alpha \frac{x-c_2}{x-c_1}$, $\alpha \in l$, thus

$$(4.13) \quad \tau_\sigma : v \mapsto \alpha \frac{x-c_2}{x-c_1} \frac{1}{v}$$

(if $c_1 = \infty$, $\varphi(x) = \alpha(x - c_2)$).

Similar discussion shows that if r were odd, there would exist two irreducible curves Γ_1, Γ_2 whose classes are $F' - \frac{r+1}{2}F$ and $F' - \frac{r-1}{2}F - E_1 - E_2$. Let v be the ratio of the defining equations of Γ_1 and Γ_2 , then we would have (4.12), but this time $\varphi(x)$ has two zeros and no pole. Since for any rational function, the number of zeros are equal with the number of poles (including $x = \infty$), this is impossible. Thus r can never be odd. \square

Let \mathfrak{G}_0 be the subgroup of \mathfrak{G} which acts trivially on $\text{Pic}(Y)$. Let $\overline{\mathfrak{G}} = \mathfrak{G}/\mathfrak{G}_0$. Then $\text{Pic}(Y)$ is actually a $\overline{\mathfrak{G}}$ -lattice. Since both of Γ_1, Γ_2 are \mathfrak{G}_0 -invariant, v is also \mathfrak{G}_0 -invariant, so we can consider only the $\overline{\mathfrak{G}}$ -action on v .

Proposition 4.18. *When $c_1, c_2 \in k \cup \{\infty\}$, $k(x, y, z)$ is k -rational.*

Proof. In this case, c_1 can never be moved by any $\sigma \in \mathfrak{G}$, so $|\overline{\mathfrak{G}}| = 2$ and the only non-trivial element of $\overline{\mathfrak{G}}$ is the above σ . Thus it suffices to find a σ -invariant transcendental basis.

Obviously $w_1 = v + \alpha \frac{x-c_2}{x-c_1} \frac{1}{v}$ and $w_2 = \sqrt{\pi_1} \{v - \alpha \frac{x-c_2}{x-c_1} \frac{1}{v}\}$ are σ -invariant. Here π_1 is one of $P(c_1), Q(c_1), -P/Q(c_1), p_0, q_0$ or $-p_0/q_0$ according to the situation, and $(x = c_1)$ is an exceptional curve of τ_σ if and only if σ maps $\sqrt{\pi_1}$ to $-\sqrt{\pi_1}$.

However, we have $l(x, y, z) = l(x, v) = l(\alpha \frac{x-c_2}{x-c_1}, v) = l(\alpha \frac{x-c_2}{x-c_1} \frac{1}{v}, v) = l(w_1, w_2)$. Thus (w_1, w_2) is a transcendental basis, and the rationality of $k(x, y, z)$ has been proved. \square

Remark 4.3. Both of $C : v_1 = \text{const.}$ and $C' : v_2 = \text{const.}$ are in the class $-F - \Omega$ as shown below.

The defining equations of C and C' are linear combinations of $(x - c_2)f_1f_2$, $(x - c_2)f_1^2$ and $(x - c_1)f_2^2$ whose classes are $2F' + (1 - r)F - E_1 - 2E_2$, $2F' + (1 - r)F - 2E_1 - E_2$ and $2F' + (1 - r)F - E_1 - 2E_2$ respectively. So the class of non-trivial linear combination is $2F' + (1 - r)F - E_1 - E_2 = -\Omega - F$.

From this we see that $C \cdot C = C' \cdot C' = C \cdot C' = 2$, $\Omega \cdot C = \Omega \cdot C' = -4$. Consider three point blow-up of Y . If $C \cdot E'_1 = C' \cdot E'_1 = C \cdot E'_2 = C' \cdot E'_3 = 1$ and $C \cdot E'_3 = C' \cdot E'_2 = 0$, then we reach $C \cdot C = C' \cdot C' = 0$, $C \cdot C' = 1$, $\Omega \cdot C = \Omega \cdot C' = -2$ on the blown-up surface Z' .

When c_1, c_2 are mutually conjugate (namely when $(x - c_1)(x - c_2)$ is irreducible over k), the analysis is more complicated. This time, $|\overline{\mathfrak{G}}| = 4$ and the non-trivial elements of $\overline{\mathfrak{G}}$ is given as follows:

$$(4.14) \quad \begin{aligned} \sigma_1 : & E_1 \mapsto F - E_1, \quad E_2 \mapsto F - E_2, \\ \sigma_2 : & E_1 \leftrightarrow E_2, \\ \sigma_3 : & E_1 \mapsto F - E_2, \quad E_2 \mapsto F - E_1. \end{aligned}$$

Note that n_σ is even as stated in Subsection 4.7.

The fixed field of \mathfrak{G}_0 is $l = k(c, \sqrt{\pi_1})$ and $\text{Gal}(l/k) = \overline{\mathfrak{G}} \simeq C_2 \times C_2$. Let k_i be the fixed field of σ_i ($i = 1, 2, 3$). They are quadratic extensions of k contained in l . Write k_i as $k_i = k(\sqrt{e_i})$, $e_i \in k$. Then $e_1 = (c_1 - c_2)^2 = (c_1 + c_2)^2 - 4c_1c_2$. l is a vector space over k with the basis $1, \sqrt{e_1}, \sqrt{e_2}$ and $\sqrt{e_3}$. Since σ_1 and σ_3 maps $\sqrt{\pi_1}$ to $-\sqrt{\pi_1}$, we have $\sqrt{\pi_1} \in k\sqrt{e_2}$ so that we can set $e_2 = \pi_1$ (especially $\pi_1 \in k$), and $e_3 = e_1e_2$.

By similar discussions as deriving (4.13) in Proposition 4.17, we can determine the action of $\overline{\mathfrak{G}}$ on v as follows:

$$(4.15) \quad \begin{aligned} \tau_{\sigma_1} : v &\mapsto \alpha \frac{x-c_2}{x-c_1} \frac{1}{v}, & \alpha \in k_1, \\ \tau_{\sigma_2} : v &\mapsto \beta \frac{1}{v}, & \beta \in k_2, \\ \tau_{\sigma_3} : v &\mapsto \gamma \frac{x-c_1}{x-c_2} v, & \gamma \gamma^{\sigma_3} = 1. \end{aligned}$$

The problem is to find a $\overline{\mathfrak{G}}$ -invariant transcendental basis of $l(x, v)$. By a suitable constant multiplication of v , we can set $\gamma = 1$. Then from $\sigma_3 = \sigma_1\sigma_2 = \sigma_2\sigma_1$, we have $\alpha = \beta \in k$. So, we set $\alpha = \beta = \kappa \in k$ and $\gamma = 1$ in (4.15).

Proposition 4.19. *$k(x, y, z)$ is k -rational if and only if $\kappa \in N_{k_1/k}(k_1)N_{k_2/k}(k_2)$.*

Proof. We see that $(x - c_1)v$ is σ_3 -invariant and is mapped to $\kappa \frac{x-c_2}{v}$ by σ_1 and σ_2 . So

$$(4.16) \quad \begin{aligned} w_1 &= (x - c_1)v + \frac{\kappa(x - c_2)}{v}, \\ w_2 &= \frac{1}{\sqrt{e_3}} \left\{ (x - c_1)v - \frac{\kappa(x - c_2)}{v} \right\} \end{aligned}$$

are $\overline{\mathfrak{G}}$ -invariant. Note that $\sqrt{e_3}$ is σ_3 -invariant and is mapped to $-\sqrt{e_3}$ by σ_1 and σ_2 .

We shall eliminate v from (4.16), then we obtain $w_1^2 - e_3 w_2^2 = \kappa(x'^2 - e_1)$ where $x' = 2x - c_1 - c_2$. From this we see that $k(x', w_1, w_2)$ is rational if and only if the corresponding quadratic form has non-zero solution in k , namely if and only if $\kappa \in N_{k_1/k}(k_1)N_{k_2/k}(k_2)$. \square

Proposition 4.20. *$k(x, y, z)$ is k -rational if and only if it is k_2 -rational, where $k_2 = k(\sqrt{\pi_1})$.*

Remark 4.4. Since $s = 0$ over k_2 , $k(x, y, z)$ is k_2 -rational except the following case.

Both of $\deg P$ and $\deg Q$ are even and $a^2 p_0 + b^2 q_0 = c^2$ has non-zero solution in $k(\sqrt{\pi_1})$ (note that the above k_2 is denoted by k_1 in Theorem 1.3).

Proof. The action of σ_2 depends only on v , independent of x , so $k(x, y, z)$ is k_2 -rational if and only if $\beta = \lambda \lambda^{\sigma_2}$ for some $\lambda \in l$. This means that by a suitable constant multiplication of v , we can set $\beta = 1$.

When $\beta = 1$, from $\sigma_3 = \sigma_1\sigma_2 = \sigma_2\sigma_1$, we have $\gamma = \frac{1}{\alpha} = \alpha^{\sigma_3}$. Then by a multiplication of v by $\gamma + 1$, γ is reduced to 1 and $\beta = 1$ is reduced to $\kappa = N_{k_1/k}(\frac{1}{\alpha} + 1) \in N_{k_1/k}(k_1)$. The proof is completed by virtue of Proposition 4.19. \square

4.10. The case where $s = 3$. Similar discussions as in the proof of Proposition 4.17 show the following:

Proposition 4.21. *There exist four irreducible curves Γ such that $\Gamma \cdot \Gamma < 0$ and $\Gamma \cdot F = 1$. When r is even, they are $\Gamma_1 \equiv F' - \frac{r}{2}F - E_1$, $\Gamma_2 \equiv F' - \frac{r}{2}F - E_2$, $\Gamma_3 \equiv F' - \frac{r}{2}F - E_3$ and $\Gamma_4 \equiv F' + (1 - \frac{r}{2})F - E_1 - E_2 - E_3$.*

When r is odd, they are $\Gamma_1 \equiv F' - \frac{r-1}{2}F - E_2 - E_3$, $\Gamma_2 \equiv F' - \frac{r-1}{2}F - E_1 - E_3$, $\Gamma_3 \equiv F' - \frac{r-1}{2}F - E_1 - E_2$ and $\Gamma_4 \equiv F' - \frac{r+1}{2}F$.

They are mutually conjugate by the action of $\overline{\mathfrak{G}}$ and satisfy $\Gamma_i \cdot \Gamma_i = \Gamma_i \cdot \Omega = -1$ and $\Gamma_i \cdot \Gamma_j = 0$ ($i \neq j$).

The action of $\overline{\mathfrak{G}}$ induces a permutation of Γ_i , we can blow down Y by Γ_i and $\overline{\mathfrak{G}}$ still acts on the blown down surface Y' in a Zariski homeomorphic way. Since $\Omega'_Y \cdot \Omega'_Y = 5 + 4 = 9$, Y' is biregular with the projective plane \mathbb{P}^2 . So $\overline{\mathfrak{G}}$ acts on \mathbb{P}^2 in a Zariski homeomorphic way, but a biregular transformation of \mathbb{P}^2 is nothing but a linear transformation of the homogeneous coordinate (ξ, η, ζ) , since $\text{Aut}(\mathbb{P}^2) \simeq \text{PGL}(3, \overline{k})$.

Each of Γ_i is mapped to a point P_j on \mathbb{P}^2 ($1 \leq j \leq 4$). By checking the intersection form, we see that $\{P_j\}$ is not colinear (= any three points do not lie on the same line), so that we can choose P_1 as $(1, 0, 0)$, P_2 as $(0, 1, 0)$, P_3 as $(0, 0, 1)$ and P_4 as $(1, 1, 1)$.

Proposition 4.22. *When $s = 3$, $k(x, y, z)$ is k -rational.*

Proof. We shall divide the proof into three subcases.

(I) The case where all $c_i \in k$ or ∞ ($1 \leq i \leq 3$).

In this case, $|\overline{\mathfrak{G}}| = 4$ and three non-trivial elements of $\overline{\mathfrak{G}}$ are as follows:

$$\begin{aligned} \sigma_1 : & E_1 \mapsto E_1, E_2 \mapsto F - E_2, E_3 \mapsto F - E_3, \\ \sigma_2 : & E_1 \mapsto F - E_1, E_2 \mapsto E_2, E_3 \mapsto F - E_3, \\ \sigma_3 : & E_1 \mapsto F - E_1, E_2 \mapsto F - E_2, E_3 \mapsto E_3. \end{aligned}$$

The action of σ_1 switches Γ_1 and Γ_4 , and Γ_2 and Γ_3 (whether r is even or odd), so it switches P_1 and P_4 , and P_2 and P_3 on \mathbb{P}^2 . Namely σ_1 induces the permutation (14)(23) of $\{P_j\}$. Similarly σ_2 induces (24)(13) and σ_3 induces (34)(12). The biregular transformation of \mathbb{P}^2 which induces the above permutation of $\{P_j\}$ is described as follows:

$$\sigma_1 = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}.$$

Namely, the action of σ_1 is $\xi \mapsto -\xi, \eta \mapsto \zeta - \xi, \zeta \mapsto \eta - \xi$, etc.

Put $\xi' = -\xi + \eta + \zeta, \eta' = \xi - \eta + \zeta, \zeta' = \xi + \eta - \zeta$. Then ξ' is mapped to ξ' by σ_1 and mapped to $-\xi$ by σ_2 and σ_3 . The similar holds for η' and ζ' also.

On the other hand, let π_i be one of $P(c_i), Q(c_i), -\frac{Q}{P}(c_i), p_0, q_0, -\frac{q_0}{p_0}$, according to the situation. Then $\pi_i \in k \setminus k^2$ and E_i is mapped to $x = c_i$ if and only if $\sqrt{\pi_i}^\sigma = -\sqrt{\pi_i}$. So $\sqrt{\pi_1}$ is mapped to $\sqrt{\pi_1}$ by σ_1 and mapped to $-\sqrt{\pi_1}$ by σ_2 and σ_3 . Hence $\sqrt{\pi_1}\xi'$ is invariant by all $\sigma \in \overline{\mathfrak{G}}$.

Similarly $\sqrt{\pi_2}\eta'$ and $\sqrt{\pi_3}\zeta'$ are also $\overline{\mathfrak{G}}$ -invariant. Thus, $\xi'' = \sqrt{\pi_1}\xi', \eta'' = \sqrt{\pi_2}\eta', \zeta'' = \sqrt{\pi_3}\zeta'$ become a \mathfrak{G} -invariant homogeneous coordinate of \mathbb{P}^2 , so they yield a \mathfrak{G} -invariant transcendent basis (v_1, v_2) of $\overline{k}(x, u)$ by $v_1 = \xi''/\zeta'', v_2 = \eta''/\zeta''$.

(II) The case where c_1 and c_2 are conjugates, and $c_3 \in k$ or ∞ .

Let $\overline{\mathfrak{G}}_1$ be the subgroup of $\overline{\mathfrak{G}}$, trivial on $k(c_1) = k(c_2)$. We have $[\overline{\mathfrak{G}} : \overline{\mathfrak{G}}_1] = 2$, and $\overline{\mathfrak{G}}_1$ acts as mentioned in (I). But if $\sqrt{\pi_3} \in k(c_1)$, $\sqrt{\pi_3}^\sigma = -\sqrt{\pi_3}$ is impossible for $\sigma \in \overline{\mathfrak{G}}_1$, so $|\overline{\mathfrak{G}}_1| = 2, |\overline{\mathfrak{G}}| = 4$. If $\sqrt{\pi_3} \notin k(c_1)$, then $|\overline{\mathfrak{G}}_1| = 4, |\overline{\mathfrak{G}}| = 8$.
(II – 1) When $\sqrt{\pi_3} \notin k(c_1)$.

As shown in (I), $\sqrt{\pi_1}\xi', \sqrt{\pi_2}\eta', \sqrt{\pi_3}\zeta'$ are $\overline{\mathfrak{G}}_1$ -invariant. $\overline{\mathfrak{G}} \simeq D_4$ and contains $\sigma_4 : E_1 \mapsto E_2, E_2 \mapsto E_1, E_3 \mapsto E_3$, which induces the permutation (12) of $\{P_j\}$. The corresponding biregular transformation of \mathbb{P}^2 is described by $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, namely

σ_4 maps as $\xi \rightarrow \eta, \eta \mapsto \xi, \zeta \mapsto \zeta$.

σ_4 maps as $\xi' \rightarrow \eta', \eta' \rightarrow \xi', \zeta' \rightarrow \zeta'$. On the other hand, since $\sigma_4 \notin \overline{\mathfrak{G}}_1$ and is of order 2, σ_4 maps $\sqrt{\pi_1}$ to $\sqrt{\pi_2}$, $\sqrt{\pi_2}$ to $\sqrt{\pi_1}$, or $\sqrt{\pi_1}$ to $-\sqrt{\pi_2}$, $\sqrt{\pi_2}$ to $-\sqrt{\pi_1}$.

Suppose that $\sqrt{\pi_1} \mapsto \sqrt{\pi_2}, \sqrt{\pi_2} \mapsto \sqrt{\pi_1}$. Then the action of σ_4 switches $\sqrt{\pi_1}\xi'$ and $\sqrt{\pi_2}\eta'$, so $\sqrt{\pi_1}\xi' + \sqrt{\pi_2}\eta'$ and $(c_1 - c_2)(\sqrt{\pi_1}\xi' - \sqrt{\pi_2}\eta')$ are $\overline{\mathfrak{G}}$ -invariant (If $\sqrt{\pi_1} \mapsto -\sqrt{\pi_2}, \sqrt{\pi_2} \mapsto -\sqrt{\pi_1}$, then it suffices only to replace $\sqrt{\pi_2}$ with $-\sqrt{\pi_2}$).

(II – 2) When $\sqrt{\pi_3} \in k(c_1)$.

In this case, $\overline{\mathfrak{G}}$ contains $\sigma_5 : E_1 \mapsto E_2, E_2 \mapsto F - E_1, E_3 \mapsto F - E_3$, so that $\overline{\mathfrak{G}}$ is a cyclic group of order 4.

Since this $\overline{\mathfrak{G}}$ is a proper subgroup of $\overline{\mathfrak{G}}$ in (II – 1), the basis obtained in (II – 1) is $\overline{\mathfrak{G}}$ -invariant in this case also. Namely $\xi'' = \sqrt{\pi_1}\xi' + \sqrt{\pi_2}\eta', \eta'' = (c_1 - c_2)(\sqrt{\pi_1}\xi' - \sqrt{\pi_2}\eta'), \zeta'' = \sqrt{\pi_3}\zeta'$ become a $\overline{\mathfrak{G}}$ -invariant homogeneous coordinate of \mathbb{P}^2 , and $v_1 = \xi''/\zeta'', v_2 = \eta''/\zeta''$ is a $\overline{\mathfrak{G}}$ -invariant transcendent basis of $\overline{k}(x, u)$.

(III) The case where all c_i are mutually conjugate ($1 \leq i \leq 3$).

Let $K = k(c_1, c_2, c_3)$ be the smallest decomposition field. Let $\overline{\mathfrak{G}}_1$ be the subgroup of $\overline{\mathfrak{G}}$, trivial on K .

(III – 1) When $[K : k] = 6$ and $\sqrt{\pi_i} \notin K$ ($1 \leq i \leq 3$).

Then $\overline{\mathfrak{G}}_1$ is $\overline{\mathfrak{G}}$ in (I), and $\overline{\mathfrak{G}}/\overline{\mathfrak{G}}_1 \simeq \mathfrak{S}_3$. (So $\overline{\mathfrak{G}} \simeq \mathfrak{S}_4$). $\sqrt{\pi_1}\xi', \sqrt{\pi_2}\eta', \sqrt{\pi_3}\zeta'$ are $\overline{\mathfrak{G}}_1$ -invariant as proved in (I). A permutation of E_i induces a permutation of P_i ($1 \leq i \leq 3$), so induces a permutation of ξ', η', ζ' . Then $\sqrt{\pi_1}, \sqrt{\pi_2}, \sqrt{\pi_3}$ are also permuted in the same way. So $\sqrt{\pi_1}\xi' + \sqrt{\pi_2}\eta' + \sqrt{\pi_3}\zeta'$ is $\overline{\mathfrak{G}}$ -invariant. Similarly $c_1\sqrt{\pi_1}\xi' + c_2\sqrt{\pi_2}\eta' + c_3\sqrt{\pi_3}\zeta'$, and $c_1^2\sqrt{\pi_1}\xi' + c_2^2\sqrt{\pi_2}\eta' + c_3^2\sqrt{\pi_3}\zeta'$ are $\overline{\mathfrak{G}}$ -invariant.

(III – 2) When $[K : k] = 3$.

The proof is same as in (III – 1), except \mathfrak{S}_3 is replaced by C_3 . (So $\overline{\mathfrak{G}} \simeq \mathfrak{A}_4$).

(III – 3) When $[K : k] = 6$ and $\sqrt{\pi_i} \in K$.

Then $|\overline{\mathfrak{G}}| = 6, \overline{\mathfrak{G}} \simeq \mathfrak{S}_3$. σ_5 in (II – 2) is contained in $\overline{\mathfrak{G}}$. The order of σ_5 is 4. This contradicts with $|\overline{\mathfrak{G}}| = 6$. Therefore this case can never happen (under the assumption $s = 3$).

□

Remark 4.5. Both of $C : v_1 = \text{const.}$ and $C' : v_2 = \text{const.}$ are in the class $-F - \Omega$ as shown below.

Let $h_i(x, u) = f_i(x)u + g_i(x)$ be the defining polynomials of Γ_i .

When r is even, $\xi = 0$ on \mathbb{P}^2 decomposes into three curves $x = c_1, \Gamma_2, \Gamma_3$ on Y , therefore $\xi = a(x - c_1)h_2h_3$ with some $a \in \overline{k}$. Similarly $\eta = b(x - c_2)h_1h_3, \zeta = c(x - c_3)h_1h_2$. The defining equations of C and C' are linear combinations of $(x - c_1)h_2h_3, (x - c_2)h_1h_3, (x - c_3)h_1h_2$ whose class is all $2F' + (1 - r)F - E_1 - E_2 - E_3 = -\Omega - F$.

When r is odd, $\xi = 0$ on \mathbb{P}^2 decomposes into three curves E_1, Γ_2, Γ_3 on Y , therefore $\xi = ah_2h_3$. Similarly $\eta = bh_1h_3, \zeta = ch_1h_2$. The defining equations of C and C' are linear combinations of h_2h_3, h_1h_3, h_1h_2 whose classes are $2F' + (1-r)F - 2E_1 - E_2 - E_3, 2F' + (1-r)F - E_1 - 2E_2 - E_3, 2F' + (1-r)F - E_1 - E_2 - 2E_3$ respectively. A non-trivial linear combination is in the class $-\Omega - F$.

From this we see that $C \cdot C = C' \cdot C' = C \cdot C' = 1, \Omega \cdot C = \Omega \cdot C' = -3$. Consider two point blow-up of Y . If $C \cdot E'_1 = C' \cdot E'_2 = 1$ and $C \cdot E'_2 = C' \cdot E'_1 = 0$, then we reach $C \cdot C = C' \cdot C' = 0, C \cdot C' = 1, \Omega \cdot C = \Omega \cdot C' = -2$ on the blown up surface Z' .

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